There is no stationary cyclically monotone Poisson matching in 2D

Francesco Mattesini

WWU Münster & MPI Leipzig

Optimal Transport and Uncertainty

Pisa, 26 November 2021

Joint with M. Huesmann (WWU Münster)
and F. Otto (MPI Leipzig)

The random Euclidean bipartite matching problem

Let X_i, Y_i be i. i. d. uniformly distributed points $(X_i)_{i=1}^n, (Y_i)_{i=1}^n$ on $(0,1)^d$ and consider the optimization problem

$$\min_{\sigma \in S_n} \sum_{i=1}^n \left| X_i - Y_{\sigma(i)} \right|^\gamma .$$

Define the empirical measures

$$\mu_n = \sum_{i=1}^n \delta_{X_i}, \quad \nu_n = \sum_{i=1}^n \delta_{Y_i}.$$

Recall the Wasserstein distance:

$$W_\gamma^\gamma\left(\mu_n, \nu_n\right) = \inf_{\pi \in \mathsf{Cpl}} \int |x-y|^\gamma \ d\pi\left(x,y\right),$$

where Cpl is the set of coupling between μ_n and ν_n, namely the set of measures on the product space with first marginal equals to μ_n and second marginal equals to ν_n.

By Birkhoff's theorem

$$W_\gamma^\gamma(\mu_n, \nu_n) = \min_{\sigma \in S_n} \sum_{i=1}^n \left| X_i - Y_{\sigma(i)} \right|^\gamma .$$

Macroscopic Behaviour

Idea: The typical distance is $n^{-\frac{1}{d}}$ (points are spread as in a regular grid)

$$\min_{\sigma \in S_n} \sum_{i=1}^{n} \left| X_i - Y_{\sigma(i)} \right|^{\gamma} \approx n \cdot n^{-\frac{\gamma}{d}}.$$

Achtung: There are fluctuations! (CLT)

The asymptotic behaviour of $W_\gamma(\mu_n, \nu_n)$ depends on the dimension d:

$$W_\gamma^\gamma(\mu_n, \nu_n) \sim \begin{cases} n \cdot n^{-\frac{\gamma}{2}} & \text{for } d = 1, \\ n \cdot \left(\frac{\log n}{n}\right)^{\frac{\gamma}{2}} & \text{for } d = 2, \\ n \cdot n^{-\frac{\gamma}{d}} & \text{for } d \geq 3. \end{cases}$$

The critical dimension $d = 2$ has been firstly understood in the seminal paper by Ajtai, Komlós and Tusnády (1983).

Topics

Question: Convergence of the rescaled cost.
Ambrosio-Stra-Trevisan (2016 $d = 2, \gamma = 2$)
Goldman-Trevisan (2020 $d \geq 3, \gamma \geq 1$)
Ambrosio-Goldman-Trevisan (2021 Lipschitz domain $d = 2, \gamma = 2$)
Bobkov-Ledoux (2019 $d = 1$)

Question: Mesoscopic behaviour.
del Barrio-Giné-Utzet (2005)
Ledoux (2019)
Goldman-Huesmann (2021)

Question: Thermodynamic limit.
Huesmann-Sturm (2013 $d \geq 3$)
Holroyd-Janson-Wästlund (2020)
$\looparrowright$ Today

The Poisson Point Process

The Poisson point process on $\mathbb{R}^d$ can be defined as a random variable taking values on locally finite atomic measures

$$\mu = \sum_i \delta_{X_i}$$

such that for every $k \geq 1$, for any disjoint Borel sets $A_1, \ldots, A_k \subseteq \mathbb{R}^d$,

- $\rightsquigarrow$ the random variables $\mu(A_1), \ldots, \mu(A_k)$ are independent,
- $\rightsquigarrow$ the random variable $\mu(A_i)$ has a Poisson distribution of parameter $|A_i|$ for every $i = 1, \ldots, k$.

Existence: Superposition argument. On $\Omega \subseteq \mathbb{R}^d$ bounded, conditionally on $\mu(\Omega) = n$, the measure $\mu \llcorner \Omega$ has the same law as the random measure

$$\sum_{i=1}^n \delta_{X_i},$$

where $(X_i)_{i=1}^n$ are i. i. d. with uniform law on Ω.

Notation: I will often make use of the notation $\{X\}$ to denote a Poisson point process.

Motivation

Definition: Consider two Poisson point process $\{X\}, \{Y\}$ in $\mathbb{R}^d$ and let T be a bijection from $\{X\}$ to $\{Y\}$. We call matching the triple $(\{X\}, \{Y\}, T)$.

Definition: A matching $(\{X\}, \{Y\}, T)$ is γ-minimal if for any finite subset $\{X_i\}_{i=1}^n \subset \{X\}, \{Y_i\}_{i=1}^n \subset \{Y\}$

$$\sum_{i=1}^n |T(X_i) - X_i|^\gamma = \min_{\sigma \in S_n} \sum_{i=1}^n \left| X_i - Y_{\sigma(i)} \right|^\gamma .$$

Question: (Peres 2002) For $\{X\}, \{Y\}$ independent Poisson processes of intensity 1 in $\mathbb{R}^2$, does there exist a stationary *planar* matching?

Question: (Holroyd 2009) For $\{X\}, \{Y\}$ independent Poisson processes of intensity 1 in $\mathbb{R}^2$, does there exist a stationary γ-*minimal* matching?

Theorem (Holroyd-Janson-Wästlund 2020)

There exists a stationary γ-minimal matching if

- $\rightsquigarrow$ $d = 2, \gamma < 1$;
- $\rightsquigarrow$ $d \geq 3, \gamma < \infty$.

Simulations

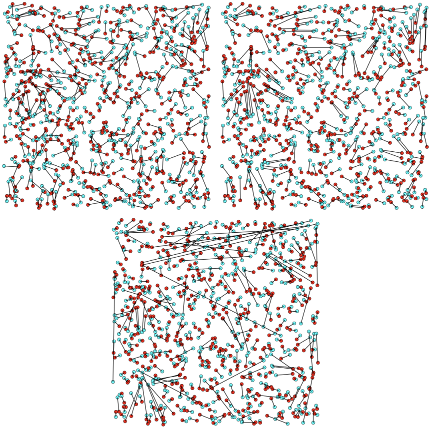


Figure: Uniformly random red and blue points in equal numbers on a square, together with γ-minimal matchings for $\gamma = \infty$ (top-left), $\gamma = 1$ (top-right), and $\gamma = -\infty$ (bottom).
Credits to Holroyd-Janson-Wästlund 2020.

Our result

We are interested in the case $\gamma = 2$, $d = 2$.

Theorem (Huesmann-M.-Otto)

There exists no stationary, ergodic and 2-minimal matching $(\{X\}, \{Y\}, T)$ in $d = 2$.

Idea of the proof: Contradiction argument. We argue by showing that 2-minimality together with stationarity will imply the following contradiction:

$$O(\ln^{\frac{1}{2}} R) \leq \frac{1}{R^d} \sum_{X \in B_R \text{ or } T(X) \in B_R} |T(X) - X| \leq o(\ln^{\frac{1}{2}} R).$$

Proof in a nutshell

Step 1: Ergodic estimate

$$\# \left\{ X \in (-R,R)^d \;:\; |T(X) - X| \gg 1 \right\} \leq o(R^d).$$

Step 2: L^∞-estimate

$$|T(X) - X| \leq o(R) \text{ provided that } X \in (-R,R)^d,$$

Step 3: Harmonic approximation

$$\frac{1}{R^d} \sum_{X \in B_R \text{ or } T(X) \in B_R} |T(X) - X|^2 \leq O(\ln R).$$

Step 4: Trading integrability against asymptotics.

$$\frac{1}{R^d} \sum_{X \in B_R \text{ or } T(X) \in B_R} |T(X) - X| \leq o(\ln^{\frac{1}{2}} R).$$

Step 5: Lower bound

$$\frac{1}{R^d} \sum_{X \in B_R \text{ or } T(X) \in B_R} |T(X) - X| \geq O(\ln^{\frac{1}{2}} R).$$

Step 1: Ergodic estimate

Definition: a matching $(\{X\}, \{Y\}, T)$ is said to be stationary if the joint law of $(\{X\}, \{Y\}, T)$ is invariant under the action of the additive group $\mathbb{Z}^d$

$$(\{X\},\{Y\},T) \mapsto (\{x+X\},\{x+Y\},T(\cdot - x)+x) \quad x \in \mathbb{Z}^d.$$

Lemma

For any $\varepsilon > 0$ there exists a deterministic L and a random radius $r_* < \infty$ a. s. such that for all $R \geq r_*$

$$\#\left\{X \in (-R,R)^2 \mid |T(X) - X| > L\right\} \leq (\varepsilon R)^2 .$$

Idea of the proof: Use stationarity and ergodicity to apply Birkhoff-von Neumann's theorem to get

$$\begin{aligned} &\frac{1}{R^d}\#\left\{X \in (-R,R)^2 \mid |T(X) - X| > L\right\} \\ &\to \mathbb{E}\left[\#\left\{X \in (-1,1)^2 \mid |T(X) - X| > L\right\}\right], \end{aligned}$$

as $R \to \infty$.

Step 2: L^∞-estimate

Lemma

For every $\varepsilon > 0$ there exists a random radius $r_* < \infty$ a. s. such that for every $R \geq r_*$

$$|T(X) - X| \leq \varepsilon R \quad \textit{provided that } X \in (-R, R)^2.$$

Key ingredient: Minimality implies that T is cyclically monotone, and in particular monotone, thus for $\{X_i\}_{i=1}^3$ we have

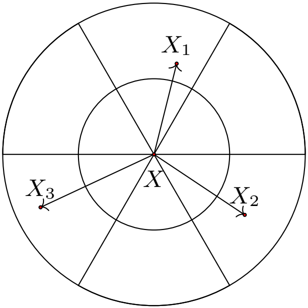
$$(T(X_i) - T(X)) \cdot (X_i - X) \geq 0.$$

In particular,

$$\begin{aligned}(T(X) - X) \cdot (X_i - X) &\leq (T(X_i) - X_i) \cdot (X_i - X) + |X_i - X|^2 \\ &\lesssim |T(X_i) - X_i|^2 + |X_i - X|^2.\end{aligned}$$

Sketch of the proof

Idea: There are enough "good" points around X.



$$(T(X) - X) \cdot \frac{(X_i - X)}{|X_i - X|} \lesssim \frac{|T(X_i) - X_i|^2}{|X_i - X|} + |X_i - X| \lesssim \varepsilon R.$$

Step 3: Harmonic approximation

Aim: Improve the L^∞-estimate to the L^2-estimate of the local energy $E(R) \leq O(\ln R)$.

Lemma

There exist a constant C and a random radius $r_* < \infty$ a. s. such that for every $R \geq r_*$ we have

$$E(R) = \frac{1}{R^d} \sum_{X \in B_R \text{ or } T(X) \in B_R} |T(X) - X|^2 \leq C \ln R.$$

Harmonic Approximation Theorem

Define the local energy

$$E(R) := \frac{1}{R^d} \sum_{X \in B_R \text{ or } T(X) \in B_R} |T(X) - X|^2 .$$

Call $\mu = \sum_{X \in (-R,R)^2} \delta_X$ and $\mu = \sum_{Y \in (-R,R)^2} \delta_Y$ define the data term

$$D(R) := \frac{1}{R^d} W^2_{(-R,R)^2}(\mu, n_\mu) + \frac{R^2}{n_\mu}(n_\mu - 1)^2 + \frac{1}{R^d} W^2_{(-R,R)^2}(\nu, n_\nu) + \frac{R^2}{n_\nu}(n_\nu - 1)^2,$$

where $n_\mu = \frac{\#\{X \in (-R,R)^2\}}{4R^2}$, $n_\nu = \frac{\#\{Y \in (-R,R)^2\}}{4R^2}$.

Theorem (Goldman-Huesmann-Otto)

For any $0 < \tau \ll 1$, there exists an $\varepsilon := \varepsilon(\tau) > 0$ and a $C_\tau < \infty$ such that provided for some R

$$\frac{1}{R^2} E(6R) + \frac{1}{R^2} D(6R) \leq \varepsilon$$

there exists a harmonic gradient field Φ such that

$$\frac{1}{R^d} \sum_{X \in B_R \text{ or } T(X) \in B_R} |T(X) - X - \nabla\Phi(X)|^2 \leq \tau E(6R) + C_\tau D(6R),$$

$$\sup_{B_{2R}} |\nabla\Phi|^2 \leq C_\tau \left(E(6R) + D(6R)\right).$$

Application of the harmonic approximation

Idea: Splitting the sum.

Consider the contribution given by the points which are transported by large distance

$$
\begin{aligned}
&\frac{1}{R^d}\sum_{(X\in B_R \text{ or } T(X)\in B_R) \text{ and } |T(X)-X|>L_\tau} |T(X)-X|^2 \\
&\leq \frac{2}{R^d}\sum_{X\in B_R \text{ or } T(X)\in B_R} |T(X)-X-\nabla\Phi(X)|^2 \\
&+\frac{2}{R^d}\sum_{(X\in B_R \text{ or } T(X)\in B_R) \text{ and } |T(X)-X|>L_\tau} |\nabla\Phi(X)|^2 \\
&\leq 2\tau(1+C_\tau)E(6R)+2C_\tau(1+\tau)\underbrace{D(6R)}_{\lesssim \ln R}.
\end{aligned}
$$

This combines to

$$E(R)\leq \tau E(6R)+C_\tau \ln R.$$

Iteration:

$$E(R)\leq \tau^k E(6^k R)+C_\tau\sum_{l=0}^{k-1}\tau^l \ln R\leq \varepsilon(36\tau)^k R^2+C_\tau\sum_{l=0}^{k-1}\tau^l\ln R.$$

Step 4: Upper bound

Lemma

For every $\varepsilon > 0$ *there exists a random radius* $r_* < \infty$ *a. s. such that*

$$\frac{1}{R^d} \sum_{X \in B_R \text{ or } T(X) \in B_R} |T(X) - X| \leq \varepsilon \ln^{\frac{1}{2}} R.$$

Proof: We split again the sum into moderate and large transportation distance and apply Cauchy-Schwarz:

$$\begin{aligned}
&\frac{1}{R^d} \sum_{X \in B_R \text{ or } T(X) \in B_R} |T(X) - X| \\
&\leq \frac{1}{R^d} \sum_{(X \in B_R \text{ or } T(X) \in B_R) \text{ and } |T(X) - X| \leq L} |T(X) - X| \\
&+ \frac{1}{R^d} \sum_{(X \in B_R \text{ or } T(X) \in B_R) \text{ and } |T(X) - X| > L} |T(X) - X| \\
&\leq CL + \varepsilon^{\frac{1}{2}} E(R)^{\frac{1}{2}} \leq C \varepsilon^{\frac{1}{2}} \ln^{\frac{1}{2}} R.
\end{aligned}$$

Conclusions

For $\gamma = 2$, $d = 2$ we proved the following:

Theorem

There exists no stationary, ergodic and 2-minimal matching $(\{X\}, \{Y\}, T)$ in $d = 2$.

Our proof relies on the Harmonic Approximation Theorem that requires $\gamma = 2$.

Question: What if $\gamma \geq 1$?
$\rightsquigarrow$ Hope: for $\gamma > 1$ a similar argument might work.
$\rightsquigarrow$ Problem: $\gamma = 1$ requires a different argument.

The Ansatz by Caracciolo-Lucibello-Parisi-Sicuro (2014)

Write $\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}, \nu_n = \sum_{i=1}^n \frac{1}{n} \delta_{Y_i}$. The optimal coupling for $W_2^2 (\mu_n, \nu_n)$ is given by $(\nabla \psi, \mathrm{Id})_\# \nu_n$.

A formal change of variable yields $\mu_n \det \nabla^2 \psi = \nu_n$.
Since $\mu_n \approx 1$ we expect $\nabla \psi (x) = x + \nabla \varphi (x)$ so that $\nabla^2 \psi = \mathrm{Id} + \nabla^2 \varphi$.

Substitution and linearization yields

$$\mu_n (1 + \Delta \varphi) = \nu_n \quad \text{s. t. with } \mu_n \approx 1 \quad \Delta \varphi = \nu_n - \mu_n.$$

This ansatz leads to $\nabla \varphi (x) = \nabla \psi (x) - x$ thus

$$\int |\nabla \varphi|^2 \, d\nu_n \approx W_2^2 (\mu_n, \nu_n) ,$$

which gives several explicit description.
$\rightsquigarrow$ Ambrosio-Stra-Trevisa (2019), Ambrosio-Glaudo-Trevisan (2019): true on macroscopic level for cost and transport map.
$\rightsquigarrow$ Goldman-Huesmann-Otto (2021): Harmonic Approximation results. For $\nu_n =$ Leb quantitative version from macro down to micro scale.

Thank you for the attention!