Teoria dei modelli della doppia appartenenza

Rosario Mennuni
in collaborazione con Bea Adam-Day e John Howe

Università di Pisa

Seminario di Logica
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In this talk

A model-theoretic look at certain graphs arising from a non-well-founded set theory.

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Plan of the talk:

- Set-up: double-membership graphs; Anti-Foundation.
- Untameness: why these graphs are (very) wild.
- Games: how ideas from finite model theory help.

Membership graphs

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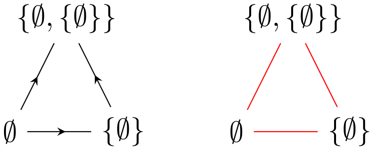

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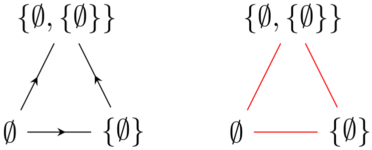

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Show that M_S satisfies the Random Graph axioms. □

How much set theory does M need? Emptyset, Pairing, Union, and **Foundation**.

Foundation: no infinite descending $\in$-sequences. In particular, no $x \in x$, no $x \in y \in x$.

What happens without Foundation?

Double-membership

Definition

Let M be an $\{\in\}$-structure. $S(x, y) :\equiv x \in y \vee y \in x$ $D(x, y) :\equiv x \in y \wedge y \in x$.

Double-membership graph M_D: reduct of M to $\{D\}$. Similarly for M_{SD}.

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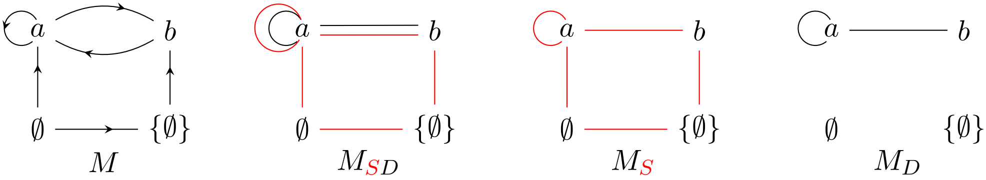

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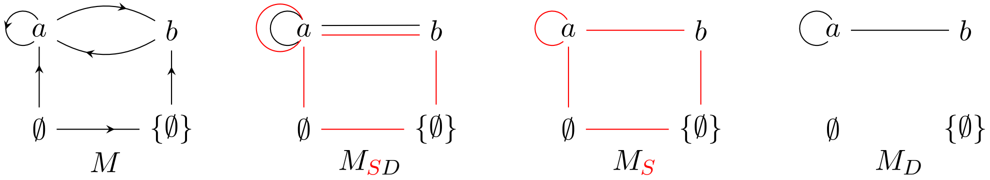

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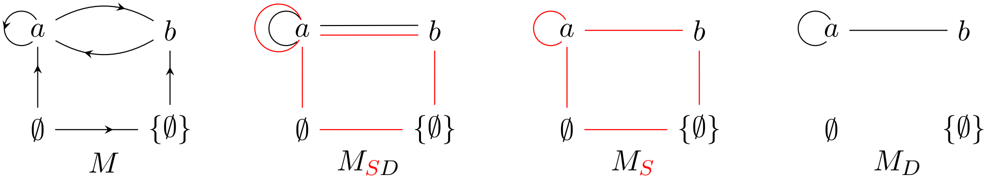

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Proposition (Adam-Day, Howe, M.)

Let G be a graph in $M \models \mathsf{ZFC}$. There is $N \models \mathsf{ZFC} \setminus \{\text{Foundation}\}$ such that N_D is isomorphic to G plus infinitely many isolated points. In particular M_S can have an arbitrary number of points with loops. Proof

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Fact (Forti, Honsell; Aczel)

ZFA is biinterpretable with (hence equiconsistent to) ZFC.

Summary of results

Starting point:

Theorem (Adam-Day, Cameron)

If $M \vDash \mathsf{ZFA}$ is countable, then M_S is the Fraïssé limit of finite loopy graphs. M_{SD} and M_D are not ω-categorical: every finite graph embeds as a union of connected components in M_D.

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Questions that were asked:

1. Are there infinitely many countable models of $\mathrm{Th}(M_{SD})$? Of $\mathrm{Th}(M_D)$?
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3. Infinite connected components of M_D?
4. ZFA with Infinity replaced by its negation?
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Fine print: assume $\mathrm{Con}(\mathsf{ZFC})$. Otherwise there might be nothing to study.

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WLOG $\operatorname{dom} G = \kappa$. Take a solution to $x_i = \{i, x_j \mid j \in \kappa, G \models R(i,j)\}(i \in \kappa)$. $\square$

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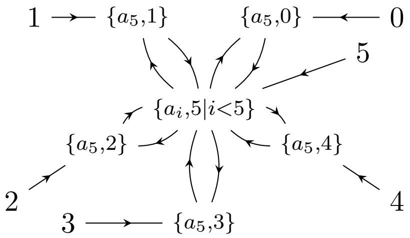

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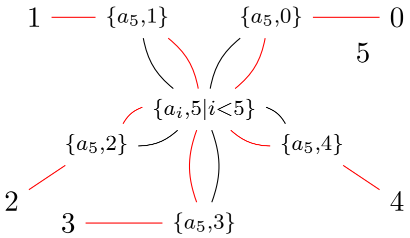

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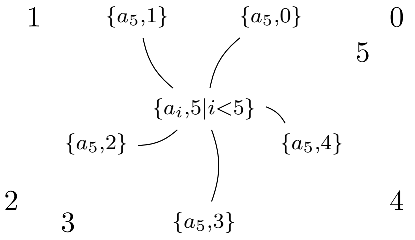

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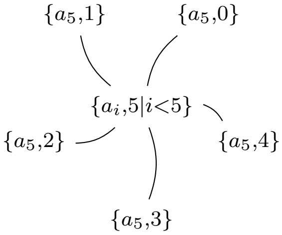

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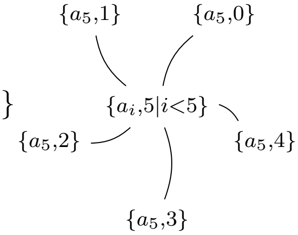

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Solutions need not be injective: if $x \mapsto a$ solves $x = \{x\}$
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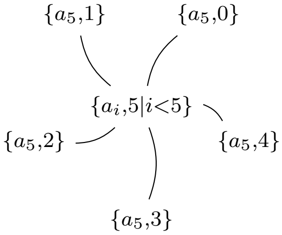

Corollary (Adam-Day, Howe, M.)

There are $2^{\aleph_0}$ countable M_D. Each of their theories has $2^{\aleph_0}$ countable models.

Proof.

For every $A \subseteq \omega \setminus \{0\}$, consider 'I have a neighbour of degree n iff $n \in A$'. $\square$

The root of all evil

It turns out that M_D is horribly complicated. This is the main reason.

Definition

Let φ be a $\{D\}$-sentence implying D is symmetric. Relativise $\exists y$ and $\forall y$ to $D(x, y)$ and call the result $\chi(x)$. Define $\mu(\varphi) := \exists x\, (\neg D(x, x) \wedge \chi(x))$.

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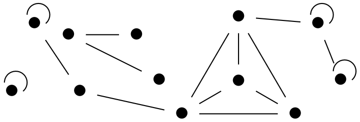
Example:  $\vDash \mu$('the universe is a triangle')

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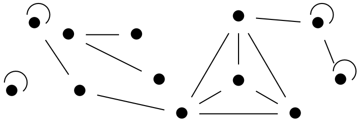

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Lemma (Adam-Day, Howe, M.)

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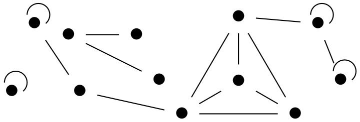
Example:  $\vDash \mu($'the universe is a triangle'$)$

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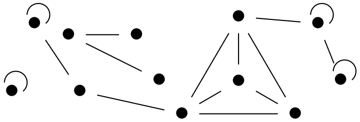
Example:  $\vDash \mu$('the universe is a triangle')

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Proof.

Add/remove a point to/from a graph and use the previous theorem. $\square$

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Proof.

1. Rosser: there is a Π^0_1 arithmetical statement independent of ZFC/ZFA.

   Rosser's Theorem=Refined version of Gödel Incompleteness.
2. Friedman–Harrington: every Π^0_1 statement is equivalent to some $\mathrm{Con}(\theta)$.
3. Translate θ into a formula φ of graphs (graphs interpret anything!).
4. Consider $\mu(\varphi)$.

$\square$

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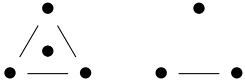

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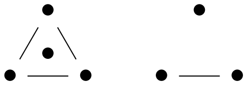

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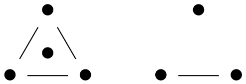

Duplicator has a winning strategy for the game of length 2; but not for the game of length 3. Same for $(\mathbb{Z}, <)$ and $(\mathbb{Q}, <)$.

Theorem (Ehrenfeucht)

Duplicator has a winning strategy iff $M \equiv_n N$ (formulas of quantifier depth n).

Reminder: Ehrenfeucht-Fraïssé games

- Two players: Spoiler and Duplicator.
- Fix *relational* structures M, N and length n of the game.
- Turn i: Spoiler plays $a_i \in M$ or $b_i \in N$, Duplicator plays in the other structure.
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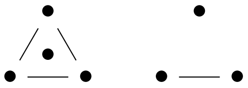

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Fact

$\equiv_n$-classes are characterised by a single formula. (The language is finite relational!)

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Theorem (Adam-Day, Howe, M.)

$A, B \models \mathrm{Th}(\{M_D \mid M \models \mathsf{ZFA}\})$. Then $A \equiv B$ iff they satisfy the same $\mu(\varphi)$'s.

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    Since M_D, N_D are actual reducts, one is free to remove the witness of $\exists$ from $\mu(\varphi)$.
- Works if natural numbers are standard. Otherwise more care is needed.

  Essentially, replace 'connected component' with 'what the model thinks is a connected component'.

$\square$

Countable nonelementarity

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Proof for M_D.

M_D has a connected component of infinite diameter. Build N as disconnected pieces satisfying the correct $\psi[1, r]$'s. Each has finite diameter. $\square$

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The same trick won't work: M_{SD} is one ball of diameter 2.

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proof of Hanf's Theorem: back-and-forth system $I_n, \dots, I_0$

$$I_j \coloneqq \{a_1, \dots, a_k \mapsto b_1, \dots, b_k \mid k \leq n - j, B((3^j - 1)/2, \bar{a}) \cong B((3^j - 1)/2, \bar{b})\}$$

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Answer.

Let N be M_{SD} without the connected components of infinite diameter.
Add a twist to the proof of Hanf's Theorem: back-and-forth system $I_n, \ldots, I_0$

$$I_j := \{a_1, \ldots, a_k \mapsto b_1, \ldots, b_k \mid k \leq n-j, B((3^j-1)/2, \bar{a}) \cong B((3^j-1)/2, \bar{b})\}$$

where the isomorphisms are in L_{SD} **but** the balls are with respect to L_D.
To show back-and-forth, write suitable flat systems in M. □

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Thanks for your attention!

Want to see what was swept under the rug?

Rieger-Bernays permutation models

Proposition (Adam-Day, Howe, M.)

Let G be a graph in $M \vDash \mathsf{ZFC}$. There is $N \vDash \mathsf{ZFC} \setminus \{\text{Foundation}\}$ such that N_D is isomorphic to G plus infinitely many isolated points. In particular M_S can have an arbitrary number of points with loops.

Proof.

WLOG $\operatorname{dom} G = \kappa$. Define $N \vDash x \in y \iff M \vDash x \in \pi(y)$, where π is the permutation swapping $a_i := \kappa \setminus \{i\}$ with $b_j := \{a_i \mid G \vDash R(i,j)\}$. Then

$$N \vDash a_i \in a_j \iff M \vDash a_i \in \pi(a_j) = b_j \iff G \vDash R(i,j)$$

and by choice of a_i and b_i there are no other D-edges.
It is an old result that $N \vDash \mathsf{ZFC} \setminus \{\text{Foundation}\}$. □

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