Modelli booleani e come fascificarli
(da un lavoro con Matteo Viale)

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Boolean algebras

Given a topological space X, let $\mathrm{CLOP}(X)$ be the boolean algebra of the clopen subsets of X.

The Stone space $\mathrm{St}(\mathsf{B})$ of a boolean algebra B is

$$\mathrm{St}(\mathsf{B}) := \{G : G \textit{ is an ultrafilter of } \mathsf{B}\}.$$

The base for the topology is:

$$\{N_b := \{G \in \mathrm{St}(\mathsf{B}) : b \in G\} : b \in \mathsf{B}\}.$$

B is isomorphic to $\mathrm{CLOP}(\mathrm{St}(\mathsf{B}))$ via the Stone duality map

$$b \mapsto N_b = \{G \in \mathrm{St}(\mathsf{B}) : \ b \in G\}$$

Boolean completions

If X is a topological space and $A \subset X$, $\mathrm{Reg}\,(A)$ is the interior of the closure of A in X. A is *regular open* if $A = \mathrm{Reg}\,(A)$.
$\mathrm{RO}(X)$ is the family of regular open subsets of X ($CLOP(X) \subseteq \mathrm{RO}(X)$).

$\mathrm{RO}(X)$ *is a complete boolean algebra, with the operations given by*

$$\neg U = X \setminus \overline{U}, \quad \bigvee_{i \in I} U_i := Reg\left(\bigcup_{i \in I} U_i\right), \quad \bigwedge_{i \in I} U_i := Reg\left(\bigcap_{i \in I} U_i\right).$$

A boolean algebra B is complete if and only if $\mathsf{CLOP}(\mathsf{St}(\mathsf{B})) = \mathsf{RO}(\mathsf{St}(\mathsf{B}))$.

Every boolean algebra B can be densely embedded in the complete boolean algebra $\mathsf{RO}(\mathsf{St}(\mathsf{B}))$ via the Stone duality map.

Boolean valued models

Definition

Let B be a *boolean algebra* and $\mathcal{L}$ be a first order *relational* language.
A B-***valued model*** for $\mathcal{L}$ is a tuple
$\mathcal{M} = \langle M, =^{\mathcal{M}}, R_i^{\mathcal{M}} : i \in I, c_j^{\mathcal{M}} : j \in J \rangle$
with

$$=^{\mathcal{M}}: M^2 \to \mathsf{B}$$
$$(\tau, \sigma) \mapsto [\![\tau = \sigma]\!]_{\mathsf{B}}^{\mathcal{M}} = [\![\tau = \sigma]\!] \,,$$

$$R^{\mathcal{M}} : M^n \to \mathsf{B}$$
$$(\tau_1, \dots, \tau_n) \mapsto [\![R(\tau_1, \dots, \tau_n)]\!]_{\mathsf{B}}^{\mathcal{M}} = [\![R(\tau_1, \dots, \tau_n)]\!]$$

for $R \in \mathcal{L}$ an n-ary relation symbol.

The constraints on $R^{\mathcal{M}}$ and $=^{\mathcal{M}}$ are the following:

- for $\tau, \sigma, \chi \in M$,
  1. $[\![\tau = \tau]\!] = 1_{\mathsf{B}}$;
  2. $[\![\tau = \sigma]\!] = [\![\sigma = \tau]\!]$;
  3. $[\![\tau = \sigma]\!] \wedge [\![\sigma = \chi]\!] \leq [\![\tau = \chi]\!]$;
- for $R \in \mathcal{L}$ with arity n, and $(\tau_1, \ldots, \tau_n), (\sigma_1, \ldots, \sigma_n) \in M^n$,

$$[\![R(\tau_1, \ldots, \tau_n)]\!] \wedge \bigwedge_{h \in \{1,\ldots,n\}} [\![\tau_h = \sigma_h]\!] \leq [\![R(\sigma_1, \ldots, \sigma_n)]\!] .$$

Definition

Let $\mathcal{M}$ be a B-valued model in the relational language $\mathcal{L}$. The *boolean value*

$$[\![\phi]\!]^{\mathcal{M}}_{\mathsf{B}} = [\![\phi]\!]$$

of the formula ϕ is defined by recursion as follows:

- $[\![\neg\psi]\!] = \neg\, [\![\psi]\!]$;
- $[\![\psi \wedge \theta]\!] = [\![\psi]\!] \wedge [\![\theta]\!]$;
- $[\![\exists y \psi(y)]\!] = \bigvee_{\tau \in M} [\![\psi(y/\tau)]\!]$.

Theorem (Soundness and completeness)

An $\mathcal{L}$-sentence ϕ is provable by an $\mathcal{L}$-theory T is and only if, for every B*-valued model for $\mathcal{L}$ in which every axiom of T has boolean value* 1*,* $[\![\phi]\!]^{\mathcal{M}}_{\mathsf{B}} = 1$.

Example 1: (ultra)products

Let $\{\mathcal{M}_x : x \in X\}$ be a family of structures for the language $\mathcal{L}$.

$$\prod_{x\in X} \mathcal{M}_x = \{f : X \to \bigcup_{x\in X} \mathcal{M}_x : f(x) \in \mathcal{M}_x\}$$

is a $\mathcal{P}(X)$-valued model for $\mathcal{L}$: if R is an n-ary relational symbol

$$\llbracket f_1 = f_2 \rrbracket = \{x \in X : f_1(x) = f_2(x)\};$$
$$\llbracket R(f_1, \dots, f_n) \rrbracket = \{x \in X : \mathcal{M}_x \vDash R(f_1(x), \dots, f_n(x))\}.$$

Example 2 (from analysis)

Let $\mathcal{M}_L$ be the algebra of Lebesgue measurable subsets of $\mathbb{R}$ and let Null be the ideal of null sets. The *measure algebra* is $\mathrm{MALG} := \mathcal{M}_L/\mathrm{Null}$.

Then $L^\infty(\mathbb{R})$ is a MALG-valued model for the language of rings $\mathcal{L} = \{+, \cdot, 0, 1\}$ where,for $f, g, h \in L^\infty(\mathbb{R})$,

$$[\![+(f, g, h)]\!] := \left[\{r \in \mathbb{R} : f(r) + g(r) = h(r)\}\right]_{\mathrm{Null}}.$$

One can prove that $L^\infty(\mathbb{R}) \vDash T_{\mathrm{fields}}$:

$$[\![\forall f (f \neq 0 \rightarrow \exists g (f \cdot g = 1))]\!] = 1_{\mathrm{MALG}}.$$

Quotients of B-valued models

Let $\mathcal{M}$ a B-valued model for $\mathcal{L}$, and F a filter over B. Consider the equivalence relation

$$\tau \equiv_F \sigma \qquad \iff \qquad [\![\tau = \sigma]\!] \in F.$$

The B$/F$-valued model $\mathcal{M}/F = \langle M/F, R_i^{\mathcal{M}/F} : i \in I, c_j^{\mathcal{M}/F} : j \in J \rangle$ is defined letting:

- $M/F := M/\equiv_F$;
- for any n-ary relation symbol R in $\mathcal{L}$

$$R^{\mathcal{M}/F}([\tau_1]_F, \dots, [\tau_n]_F) = [[\![R(\tau_1, \dots, \tau_n)]\!]]_F \in \mathrm{B}/F;$$

- For any constant symbol c in $\mathcal{L}$, $c^{\mathcal{M}/F} = [c^{\mathcal{M}}]_F$.

In particular, if G is an ultrafilter, $\mathcal{M}/G$ is a traditional first order structure.

Fullness

Definition

Given a first order signature $\mathcal{L}$, a B-valued model $\mathcal{M}$ for $\mathcal{L}$ is ***full*** if for all ultrafilters G on B, all $\mathcal{L}$-formulas $\phi(x_1, \ldots, x_n)$ and all $\tau_1, \ldots, \tau_n \in \mathcal{M}$

$$\mathcal{M}/_G \models \phi([\tau_1]_G, \ldots, [\tau_n]_G) \quad \text{if and only if} \quad [\![\phi(\tau_1, \ldots, \tau_n)]\!]^{\mathcal{M}} \in G.$$

Example

- If $\{\mathcal{M}_x : x \in X\}$ is a family of $\mathcal{L}$ structures, $\prod_{x \in X} \mathcal{M}_x$ is a full model for $\mathcal{L}$ (Łoś Theorem).
- The MALG-valued model $L^\infty(\mathbb{R})$ is not full for $\mathcal{L} = \{+, \cdot, 0, 1\}$ since we can find ultrafilters $G \in \mathrm{St}(\mathrm{MALG})$ such that $L^\infty(\mathbb{R})/G$ is not a field.

Theorem (Łoś Theorem for boolean valued models)

Let $\mathcal{M}$ be a B*-valued model for the signature $\mathcal{L}$.*
The following are equivalent:

1. *$\mathcal{M}$ is full, i.e. $\mathcal{M}/_G \models \phi([\tau_1]_G, \dots, [\tau_n]_G) \iff [\![\phi(\tau_1, \dots, \tau_n)]\!]^{\mathcal{M}} \in G$;*
2. *for all $\mathcal{L}_{\mathcal{M}}$-formulas $\phi(x_0, \dots, x_n)$ and all $\tau_1, \dots, \tau_n \in \mathcal{M}$ there exists $\sigma_1, \dots, \sigma_m \in \mathcal{M}$ such that*

$$\bigvee_{\sigma \in \mathcal{M}} [\![\phi(\sigma, \tau_1, \dots, \tau_n)]\!] = \bigvee_{i=1}^{m} [\![\phi(\sigma_i, \tau_1, \dots, \tau_n)]\!]$$

Mixing property

Definition

A $\mathbb{B}$-valued model $\mathcal{M}$ satisfies the ***mixing property*** if for every antichain $A \subset \mathbb{B}$, and for every subset $\{\tau_a : a \in A\} \subseteq M$, there exists $\tau \in M$ such that

$$a \leq [\![\tau = \tau_a]\!] \text{ for every } a \in A.$$

Proposition

Let $\mathcal{M}$ be a $\mathbb{B}$-model for $\mathcal{L}$ satisfying the mixing property. Then $\mathcal{M}$ is full.

If M is a countable transitive model of ZFC, then the forcing $\mathbb{B}$-valued model for set theory $M^{\mathbb{B}}$ is full but not mixing.

Presheaves and sheaves

For X a topological space, a X***-presheaf*** is a contravariant functor $\mathcal{O}(X) \to \mathrm{Set}$.

A X-presheaf $\mathcal{F}$ is a X***-topological sheaf*** if for every family $\{U_i : i \in I\} \subseteq \mathcal{O}(X)$ with $U = \bigcup_{i \in I} U_i$:

1. if $f, g \in \mathcal{F}(U)$ are such that $\mathcal{F}(U_i \subseteq U)(f) = \mathcal{F}(U_i \subseteq U)(g)$ then $f = g$;
2. if $\{f_i \in \mathcal{F}(U_i) : i \in I\}$ is a *matching family* i.e. such that, for $i \neq j$

   $$\mathcal{F}(U_i \cap U_j \subseteq U_i)(f_i) = \mathcal{F}(U_i \cap U_j \subseteq U_j)(f_j),$$

   then there exists a *collation* $f \in \mathcal{F}(U)$ such that

   $$\mathcal{F}(U_i \subseteq U)(f) = f_i \quad \textit{for every } i \in I.$$

A X-presheaf $\mathcal{F}$ is a X***-stonean sheaf*** if for every family $\{U_i : i \in I\} \subseteq \mathcal{O}(X)$ with $\bigcup_{i\in I} U_i$ dense in $U \in \mathcal{O}(X)$:

1. if $f, g \in \mathcal{F}(U)$ are such that $\mathcal{F}(U_i \subseteq U)(f) = \mathcal{F}(U_i \subseteq U)(g)$ then $f = g$;
2. if $\{f_i \in \mathcal{F}(U_i) : i \in I\}$ is a *matching family* i.e. such that, for $i \neq j$

$$\mathcal{F}(U_i \cap U_j \subseteq U_i)(f_i) = \mathcal{F}(U_i \cap U_j \subseteq U_j)(f_j),$$

   then there exists a *collation* $f \in \mathcal{F}(U)$ such that

$$\mathcal{F}(U_i \subseteq U)(f) = f_i \quad \textit{for every } i \in I.$$

We can replace U in the definition with $\operatorname{Reg}\left(\bigcup_{i\in I} U_i\right)$.

A stonean sheaf is also a topological sheaf.

Boolean valued models as presheaves

Given a complete boolean algebra B *and a* B*-valued model* $\mathcal{M}$*, its associated presheaf* $\mathcal{F}_{\mathcal{M}} : \mathcal{O}(\mathsf{St}(\mathsf{B}))^{op} \to \mathsf{Set}$ *is such that*

- $\mathcal{F}_{\mathcal{M}}(U) = \mathcal{M}/_{F_{Reg(U)}}$ *where* $F_{Reg(U)}$ *is the filter generated by* $Reg(U)$*;*
- $\mathcal{F}_{\mathcal{M}}(U \subseteq V)$ *is the map*

$$\begin{aligned} i^{\mathcal{M}}_{UV} : \mathcal{M}/_{F_{Reg(V)}} &\to \mathcal{M}/_{F_{Reg(U)}} \\ [\tau]_{F_{Reg(V)}} &\mapsto [\tau]_{F_{Reg(U)}}. \end{aligned}$$

Theorem (Monro - ‘86)

Let B *be a complete boolean algebra. Then the* B*-valued model* $\mathcal{M}$ *has the mixing property if and only if the presheaf* $\mathcal{F}_{\mathcal{M}}$ *is a sheaf.*

Note: $\mathcal{F}_{\mathcal{M}}$ is a topological sheaf if and only if it is a stonean sheaf.

Presheaves as boolean valued models

Given a topological space X and a X-presheaf $\mathcal{F} : \mathcal{O}(X)^{op} \to \mathrm{Set}$, we can asssociate to it a $\mathrm{RO}(X)$-valued model $\mathcal{M}_{\mathcal{F}}$ for the empty language:

$$\mathcal{M}_{\mathcal{F}} = \mathcal{F}(X)$$

with the interpretation for $=$

$$[\![f = g]\!] := Reg\Big(\bigcup \{U \in \mathcal{O}(X) : \mathcal{F}(U \subseteq X)(f) = \mathcal{F}(U \subseteq X)(g)\}\Big).$$

While $\mathcal{M} \cong \mathcal{M}_{\mathcal{F}_{\mathcal{M}}}$, $\mathcal{F}$ and $\mathcal{F}_{\mathcal{M}_{\mathcal{F}}}$ can have little in common:

- $\mathcal{F}$ is defined on X while $\mathcal{F}_{\mathcal{M}_{\mathcal{F}}}$ is defined on $\mathrm{St}(\mathrm{RO}(X))$;
- if $\mathcal{F}$ is a topological sheaf but not stonean, then $\mathcal{F}_{\mathcal{M}_{\mathcal{F}}}$ is not a (topological) sheaf.

Topological sheafification

Definition

Let X be a topological space and $\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathsf{Set}$ be a presheaf.

$$E_{\mathcal{F}} := \coprod_{x \in X} \mathcal{F}_x = \{[f]_{\sim_x} : x \in X, f \in \mathcal{F}(U), x \in U \in \mathcal{O}(X)\}$$

projects onto X via the map $p_{\mathcal{F}} : [f]_{\sim_x} \mapsto x$. Each $f \in \mathcal{F}(U)$ determines

$$\begin{aligned} \dot{f} : U &\to E_{\mathcal{F}} \\ x &\mapsto [f]_{\sim_x} \end{aligned}.$$

The topology of $E_{\mathcal{F}}$ is generated by the family of the $\dot{f}[U]$'s.

The ***topological sheafification*** $\mathrm{sh}(\mathcal{F}) : \mathcal{O}(X)^{\mathrm{op}} \to \mathsf{Set}$ is the (topological) sheaf of continuous sections of $p_{\mathcal{F}}$.

Stonean sheafification

Definition

Let $X = \mathrm{St}(\mathrm{RO}(X))$ and $\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathrm{Set}$ be a presheaf.
Its ***stonean sheafification*** is the (stonean) sheaf $\mathrm{St\text{-}sh}(\mathcal{F}) : \mathcal{O}(X)^{\mathrm{op}} \to \mathrm{Set}$ where $\mathrm{St\text{-}sh}(\mathcal{F})(U)$ is the set of continuous functions

$$s : U \to E_{\mathcal{F}} \cup \{\infty\}$$

such that

- $\{x \in U : s(x) \in E_{\mathcal{F}}\}$ contains an open dense subset D_s of U;
- $s \upharpoonright D_s$ is a section of $p_{\mathcal{F}}$ (i.e. $s \upharpoonright D_s \in \mathrm{sh}(\mathcal{F})(D_s)$).

Theorem

The functor $\mathcal{F} \mapsto \mathit{St\text{-}sh}(\mathcal{F})$ is left adjoint to the inclusion of the stonean X-sheaves into the X-presheaves.

Sheafifing a boolean valued model

Let $\mathcal{M}$ be a B-valued model for the language $\mathcal{L}$. Its sheafification is the B-valued model for the language $\mathcal{L}$

$$\mathcal{M}^+ = \{s : \mathrm{St}(\mathsf{B}) \to E_{\mathcal{F}_{\mathcal{M}}} \cup \{\infty\} : s \textit{ continuous and } s^{-1}[\{\infty\}] \textit{ nowhere dense}\}.$$

Consider the maps

$$\begin{aligned} \xi_G : & \mathcal{M}/_G \to \mathcal{M}^+/_G \\ & [\sigma]_G \mapsto [\dot{\sigma}]_G \end{aligned}.$$

Proposition

- *if $G \in \mathrm{St}(\mathsf{B})$ is generic, then ξ_G is an isomorphism;*
- *$\mathcal{M}$ has the mixing property if and only if, for every $G \in \mathrm{St}(\mathsf{B})$, ξ_G is an isomorphism;*
- *$\mathcal{M}$ is full if and only if, for every $G \in \mathrm{St}(\mathsf{B})$, ξ_G is an elementary embedding.*

An example: B-names for the real numbers

Let $\mathcal{L}$ be a relational language whose interpretation in $\mathbb{R}$ is Borel.
The family $C(\mathrm{St}(\mathsf{B}), \mathbb{R})$ is a B-valued model for $\mathcal{L}$ with the interpretation

$$[\![R(f_1, \ldots, f_n)]\!]^{C(\mathrm{St}(\mathsf{B}),\mathbb{R})} := \mathrm{Reg}\left(\{G \in \mathrm{St}(\mathsf{B}) : R^{\mathbb{R}}(f_1(G), \ldots, f_n(G))\}\right).$$

This model has not the mixing property: take a countable antichain and the family of constant functions $\{c_n : G \mapsto n\}_{n \in \mathbb{N}}$.

Its stonean sheafification is the sheaf

$$C^{+}(\mathrm{St}(\mathsf{B}), \mathbb{R}) = \{s : \mathrm{St}(\mathsf{B}) \to \mathbb{R} \cup \{\infty\} : s \textit{ continuous and } s^{-1}[\{\infty\}] \textit{ meager}\}.$$

It is isomorphic (see Vaccaro - Viale) to the B-model

$$\{\tau \in V^{\mathsf{B}} : [\![\tau \in \mathbb{R}]\!]^{V^{\mathsf{B}}} = 1\}.$$

References

- Monro. *A Category-Theoretic Approach to Boolean-Valued Models of Set Theory*. Journal of Pure and Applied Algebra 42 (1986) 245-274.
- Monro. *Quasitopoi, Logic and Heyting-Valued Models*. Journal of Pure and Applied Algebra 42 (1986) 141-164.
- Loullis. *Sheaves and Boolean Valued Model Theory*. Journal of Symbolic Logic 44.2 (1979) 153-183.
- P. - Viale. *Boolean valued models, presheaves, and étalé spaces*. arXiv: 2006.14852.
- Vaccaro - Viale. *Generic absoluteness and boolean names for elements of a Polish space*. Bollettino dell'Unione Matematica Italiana 10(3) (2017) 293–319.

THANK YOU!