Contact surface of Cheeger sets

work in collaboration with Simone Ciani (UniFi)

Marco Caroccia
Dipartimento di Matematica, Politecnico di Milano

Isoperimetric problems
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CHEEGER CONSTANT AND CHEEGER SETS

The Cheeger constant is defined, for an open bounded set $\Omega \subset \mathbb{R}^d$, as

$$h(\Omega) := \inf_{E \subseteq \Omega} \left\{ \frac{P(E)}{\mathcal{L}^d(E)} \right\}$$

being $P(E)$ the *distributional perimeter of E* (i.e. $\mathcal{H}^{d-1}(\partial E)$ for regular enough sets) and $\mathcal{L}^d(E)$ the Lebesgue measure of E.

Any set attaining

$$\frac{P(E)}{\mathcal{L}^d(E)} = h(\Omega)$$

is called a *Cheeger set of (for)* Ω.

[1](Partial) list of literature include the works of: Bucur, Buttazzo, Caselles, Cheeger, Chambolle, Figalli, Fragalà, Kawhol, Leonardi, Maggi, Neumayer, Novaga, Parini, Pratelli, Saracco, Verzini, Velichkov, and many, many others...

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The Cheeger constant of a domain is linked to the first eigenvalue of the Dirichlet p-laplacian[1]:

$$\lambda_p(\Omega) \geq \left(\frac{h(\Omega)}{p} \right)^p, \quad \lim_{p \to 1^+} \lambda_p(\Omega) = h(\Omega).$$

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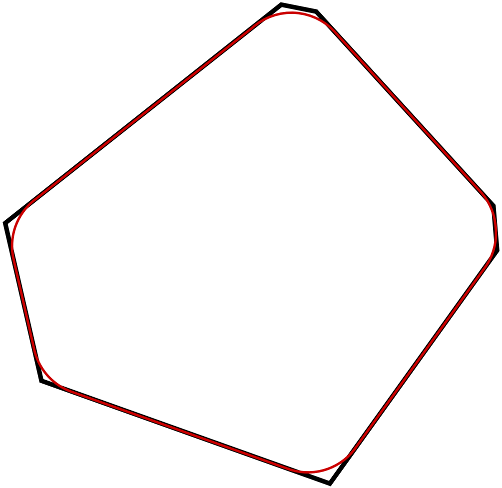
Some examples



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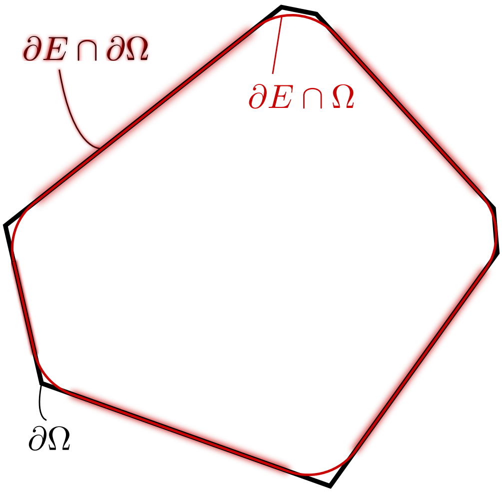

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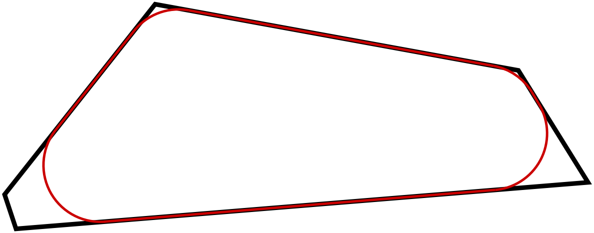
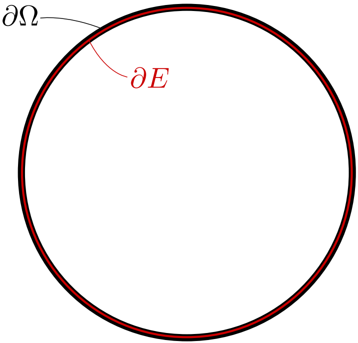


SOME EXAMPLES



Remark: the only set having a ball as a Cheeger set is the ball itself.[2]

[2] It can be viewed as a consequence of the regularity theory for the free boundary, or as a consequence of Figalli, Maggi, Pratelli: *A note on Cheeger sets*. Proceedings of the American Mathematical Society (2009): 2057-2062.

Plan of the talk

- Known properties of Cheeger sets;
- About the contact surface
    - An easy bound;
    - Main theorem: a lower bound on the (dimension of the) contact surface;
- Sketch of the proof
    - Strategy of the proof;
    - Removable singularities;
- Sharpness of the bounds

EXISTENCE AND INTERIOR REGULARITY

If $\Omega \subseteq \mathbb{R}^2$ then[3] [4]

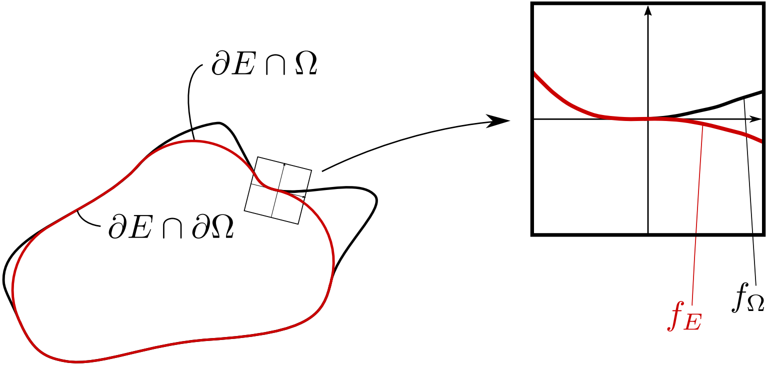
(e) there exists at least one Cheeger set E of Ω;

(ir) $\partial E \cap \Omega$ is an analytic hyper-surface with constant mean curvature equal to $h(\Omega)$;

[3] Leonardi, *An overview on the cheeger problem*. In New trends in shape optimization, pages 117–139. Springer, 2015.

[4] Parini, *An introduction to the cheeger problem.* Surv. Math. Appl., 6:9–21, 2011.

Boundary regularity



$$f_\Omega \in C^1 \;\Rightarrow\; f_E \in C^1$$
$$f_\Omega \in C^{1,1} \;\Rightarrow\; f_E \in C^{1,1}$$
$$\Omega \text{ convex} \;\Rightarrow\; f_E \in C^{1,1}.$$

BOUNDARY REGULARITY

If $\Omega \subseteq \mathbb{R}^2$ then[5]

(br.1) If E is a Cheeger set of Ω and Ω is regular enough then $\partial E \cap \Omega$ meets $\partial\Omega$ only in a tangential way, namely $\nu_E(x) = \nu_\Omega(x)$ on $\partial E \cap \partial\Omega$ [6];

(br.2) If $\partial\Omega \in C^1$ then ∂E has regularity of class C^1 in a neighbourhood of any $x \in \partial E \cap \partial\Omega$[7];

(br.3) If $\partial\Omega \in C^{1,1}$ then ∂E has regularity of class $C^{1,1}$ in a neighbourhood of any $x \in \partial E \cap \partial\Omega$[8];

(br.4) If Ω is convex then there exists a unique Cheeger set E. Moreover ∂E has regularity of class $C^{1,1}$ in a neighbourhood of any $x \in \partial E \cap \partial\Omega$[9] [10];

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[5] Gonzalez, Massari, Tamanini. *Minimal boundaries enclosing a given volume*. Manuscripta mathematica, 34(2-3):381–395, 1981.

[6] Leonardi, Pratelli, *On the cheeger sets in strips and non-convex domains*. Calculus of Variations and Partial Differential Equations, 55(1):15, 2016.

[7] Miranda, *Frontiere minimali con ostacoli*. Annali dell'Università di Ferrara, 16(1):29–37, 1971

[8] Caselles, Chambolle, Novaga, *Some remarks on uniqueness and regularity of cheeger sets*. Rend. Semin. Mat. Univ. Padova, 123:191–201, 2010

[9] Caselles, Chambolle, Novaga. *Uniqueness of the Cheeger set of a convex body*. Pacific Journal of Mathematics 232.1 (2007): 77-90.

[10] Alter, Caselles *Uniqueness of the Cheeger set of a convex body* Nonlinear Analysis: Theory, Methods and Applications 70.1 (2009): 32-44.

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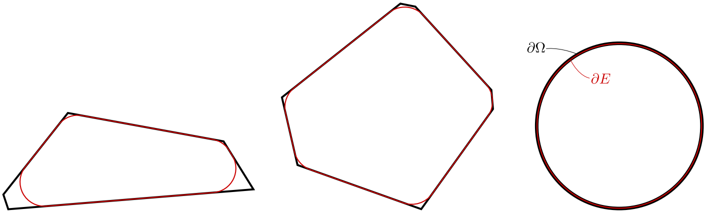

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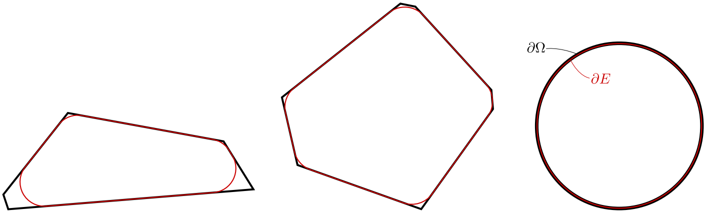

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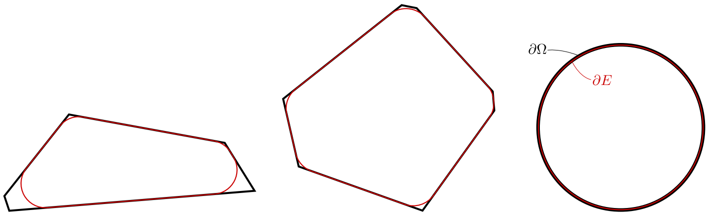
Boundary regularity



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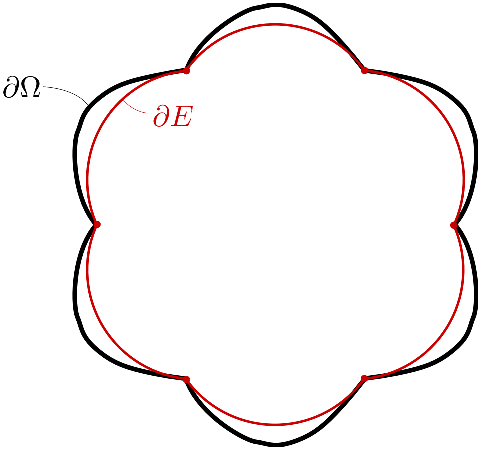
(Dr) $\mathcal{H}^{d-1}(\partial E \cap \partial\Omega) > 0$? Observe that $\mathcal{H}^{d-1}(\partial E \cap \partial\Omega)$ can be interpreted somehow as $\lim_{p\to 1} \int_{\partial\Omega} |\partial_\nu u_p|^p \, d\mathcal{H}^{d-1}$.

One more example



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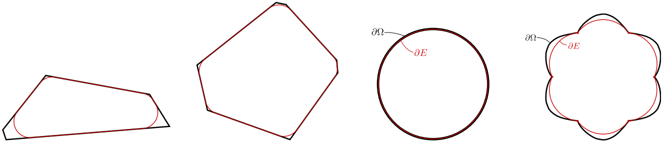

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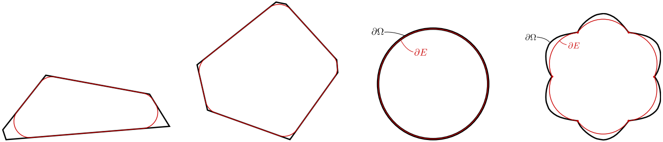

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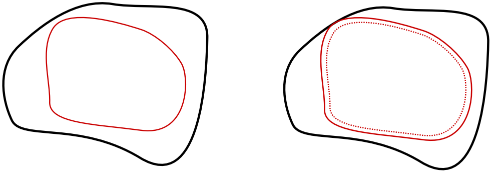
(Dr) $\mathcal{H}^{d-1}(\partial E \cap \partial\Omega) > 0$ in some cases...;

How small can $\partial E \cap \partial\Omega$ be: an easy bound



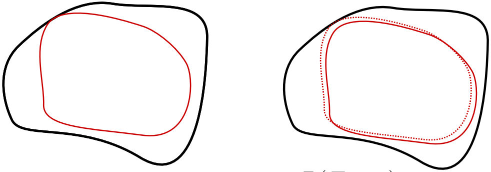
If $\mathcal{H}^0(\partial E \cap \partial\Omega) = 0$ $\qquad$ $\frac{P(\lambda E)}{\mathcal{L}^d(\lambda E)} = \frac{h(\Omega)}{\lambda} < h(\Omega)$

Suppose $\mathcal{H}^0(\partial E \cap \partial\Omega) = 0$. Then there is a $\lambda > 1$ such that $E_\lambda = \lambda E \subset \Omega$, and

$$h(\Omega) \le \frac{P(E_\lambda)}{\mathcal{L}^d(E_\lambda)} = \frac{1}{\lambda}\frac{P(E)}{\mathcal{L}^d(E)} = \frac{h(\Omega)}{\lambda} \quad \Rightarrow \quad \lambda = 1.$$

Thus $\mathcal{H}^0(\partial E \cap \partial\Omega) \ge 1.$

HOW SMALL CAN $\partial E \cap \partial\Omega$ BE: AN EASY BOUND



If $\mathcal{H}^0(\partial E \cap \partial\Omega) = 1$

$$\frac{P(E+\tau)}{\mathcal{L}^d(E+\tau)} = h(\Omega)$$

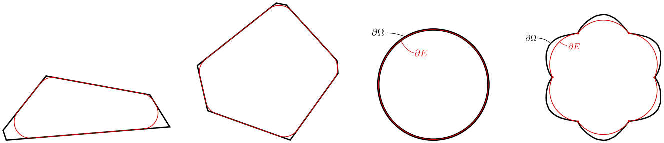
$$\mathcal{H}(\partial E \cap \partial\Omega) = 0$$

Suppose $\mathcal{H}^0(\partial E \cap \partial\Omega) = 1$. Then there is a $\tau \in \mathbb{R}^d$ such that $E_\tau = E + \tau \subset \Omega$, $\mathcal{H}^0(\partial E_\tau \cap \partial\Omega) = 0$ and

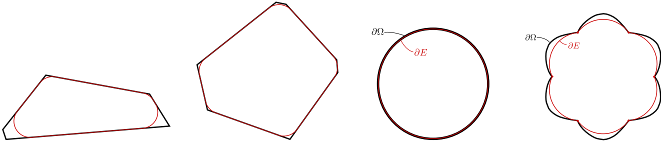
$$h(\Omega) \leq \frac{P(E_\tau)}{\mathcal{L}^d(E_\tau)} = \frac{P(E)}{\mathcal{L}^d(E)} = h(\Omega).$$

Then E_τ Cheeger set with $\mathcal{H}^0(\partial E_\tau \cap \partial\Omega) = 0$. Then the previous argument applies.

Thus $\mathcal{H}^0(\partial E \cap \partial\Omega) \geq 2$.



$$\mathcal{H}^0(\partial E \cap \partial \Omega) \geq 2$$



$$\mathcal{H}^0(\partial E \cap \partial\Omega) \geq 2$$

Theorem (C., Ciani 2020)

If $\partial\Omega$ has regularity of class $C^{1,\alpha}$, for $\alpha \in [0,1]$ then

$$\mathcal{H}^{d-2+\alpha}(\partial E \cap \partial\Omega) > 0$$

for any $E \subset \Omega$ Cheeger set. Moreover if $\alpha = 0$ then

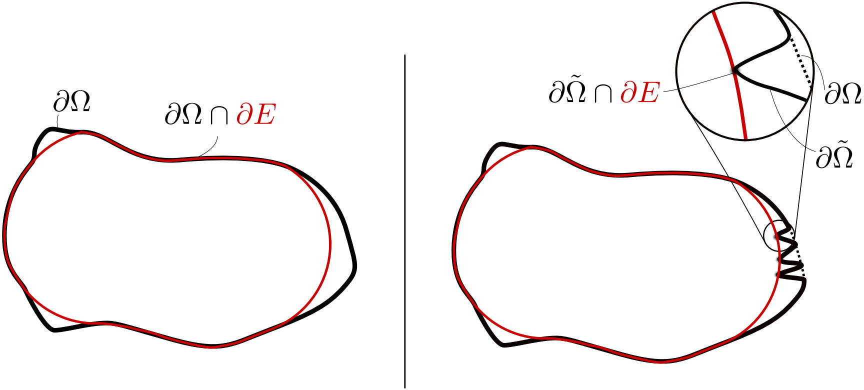
$$\mathcal{H}^{d-2}(\partial E \cap \partial\Omega) = +\infty.$$

In $d = 2$, for any $\alpha \in (0,1)$ there exists an open bounded set Ω with a Cheeger set $E \subset \Omega$, and with $\partial\Omega \in C^{1,\alpha}$, satisfying

$$\mathcal{H}^{\alpha}(\partial E \cap \partial\Omega) > 0, \;\; \mathcal{H}^{s}(\partial E \cap \partial\Omega) = 0 \;\; \textit{for any } s > \alpha.$$

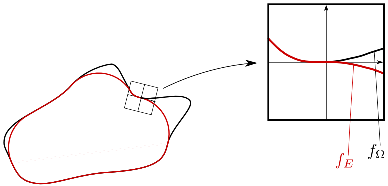
Warning

Locally the statement is false:



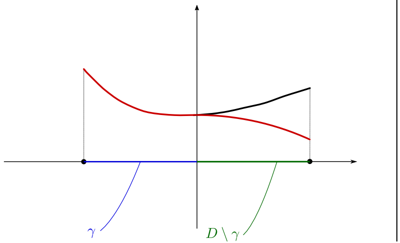
Strategy of the proof

Step 0) Assume that Ω is not a ball.



Pick $x \in \partial E \cap \partial\Omega$ (which exists since $\mathcal{H}^0(\partial E \cap \partial\Omega) > 1$).

Step 1) Let $f_E, f_\Omega : D \to \mathbb{R}$ representing $\partial E, \partial\Omega$.



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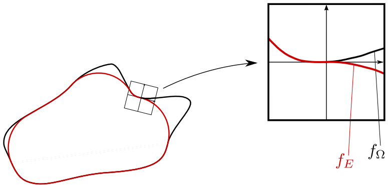

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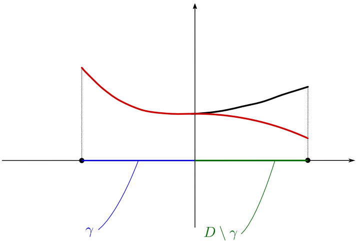


Set

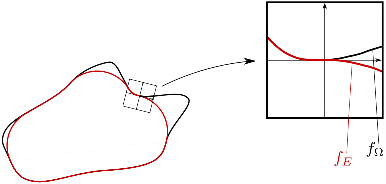
$$\gamma := \{x \in D \,|\, (x, f_E(x)) \in \partial E \cap \partial\Omega\} \subset \mathbb{R}^{d-1}$$

Then f_E satisfies

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla f_E(x)}{\sqrt{1+|\nabla f_E(x)|^2}}\right) = h(\Omega) & \text{on } D \setminus \gamma \\ f_E \leq f_\Omega & \text{on } D \end{cases}$$

STRATEGY OF THE PROOF

Step 0) Assume that Ω is not a ball.



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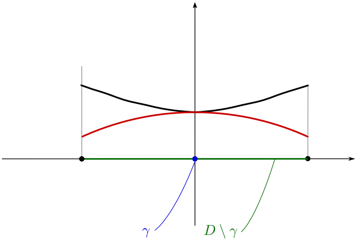

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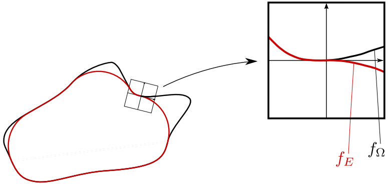

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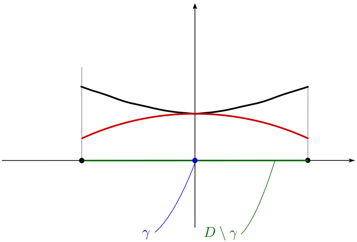

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Actually here, f_E satisfies

$$-\mathrm{div}\left(\frac{\nabla f_E(x)}{\sqrt{1+|\nabla f_E(x)|^2}}\right) = h(\Omega) \text{ on } D.$$

Strategy of the proof

$$-\mathrm{div}\left(\frac{\nabla f_E}{\sqrt{1+|\nabla f_E(x)|^2}}\right) = h(\Omega) \quad \text{on } D \setminus \gamma \tag{1}$$

Step 2): Suppose that γ is small enough to guarantee that if f_E is a solution to (1) then

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- Then ∂E has constant mean curvature in the small cube $Q_r(x)$.

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- But if $\partial E \cap \partial \Omega$ is globally small, then γ will be small around any contact point $x \in \partial E \cap \partial \Omega$.

Strategy of the proof

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- By applying the above argument on every $x \in \partial E \cap \partial\Omega$ we conclude that ∂E has constant mean curvature around any contact point.

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- Then ∂E has constant mean curvature in the small cube $Q_r(x)$.

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- By applying the above argument on every $x \in \partial E \cap \partial\Omega$ we conclude that ∂E has constant mean curvature around any contact point.

- But the free boundary is also a constant mean curvature hypersurface.

Strategy of the proof

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$$-\text{div}\left(\frac{\nabla f_E}{\sqrt{1+|\nabla f_E(x)|^2}}\right) = h(\Omega) \quad \text{on } D. \tag{2}$$

- Then ∂E has constant mean curvature in the small cube $Q_r(x)$.

- But if $\partial E \cap \partial\Omega$ is globally small, then γ will be small around any contact point $x \in \partial E \cap \partial\Omega$.

- By applying the above argument on every $x \in \partial E \cap \partial\Omega$ we conclude that ∂E has constant mean curvature around any contact point.

- But the free boundary is also a constant mean curvature hypersurface.

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STRATEGY OF THE PROOF

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- But the free boundary is also a constant mean curvature hypersurface.

Then ∂E is an hyper-surface with CMC. (Up to a small set Σ) $\Rightarrow$ Alexandrov's Theorem (revised)[11] $\Rightarrow$ E is a ball.

[11] Delgadino, Maggi. *Alexandrov's theorem revisited*. Anal. PDE (2019).

Idea of the proof

Summarizing, if $\partial E \cap \partial\Omega$ is small enough (in some sense) then E has to be a ball.

Contradiction: The ball can be a Cheeger set only of the ball! Therefore $\Omega = E$ is a ball.

IDEA OF THE PROOF

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Then $\partial E \cap \partial\Omega$ cannot be too small. At least as big as it is required so that, its pre-image γ, (somewhere) cannot be removed for the CMC equation:

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = h \ \text{ on } D \setminus \gamma \quad \not\Rightarrow \quad -\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = h \ \text{ on } D.$$

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Notice that the removability of γ has to depend in some sense on the regularity of u in D, i.e. the more regular is u the bigger γ can be.

For instance if we know a priori that $u \in C^2(D)$ then any closed set γ with $\mathcal{H}^{d-1}(\gamma) = 0$ is removable. (Recall that $u : \mathbb{R}^{d-1} \to \mathbb{R}$)

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For our purposes, we cannot rely on a regularity better than $u \in C^{1,1}(D)$

REMOVABLE SINGULARITIES

If u a solution to

$$Lu = 0 \ \text{ on } D \setminus \gamma,$$

when can we conclude also

$$Lu = 0 \ \text{ on } D?$$

Typically it depends on the regularity of $u \in \mathrm{Cl}(D)$ and on the size of γ.[12]

[12] A very partial list of literature on this topic:

- Serrin, *Isolated singularities of solutions of quasi-linear equations*. Acta Mathematica, 113:219–240, 1965.
- Serrin, *Removable singularities of solutions of elliptic equations II*. Archive for Rational Mechanics and Analysis,20(3):163–169, 1965
- De Giorgi, Stampacchia. *Sulle singolarità eliminabili delle ipersuperficie minimali* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8), 38:352–357, 1965
- De Pauw, Pfeffer. *The gauss–green theorem and removable sets for pdes in divergence form*. Advances in Mathematics, 183(1):155–182, 2004
- Simon *On a theorem of De giorgi and Stampacchia*. Mathematische Zeitschrift, 155(2):199–204, 1977.

De Giorgi, Stampacchia - Simon Theorem

Let $u \in C^2(D \setminus \gamma)$, $D \subset \mathbb{R}^{d'}$ satisfy

$$-\mathrm{div}\left(\frac{\nabla u(x)}{\sqrt{1+|\nabla u(x)|^2}}\right) = H \quad \text{for all } x \in D \setminus \gamma$$

and $\mathcal{H}^{d'-1}(\gamma) = 0$ then there exists a unique extension $\tilde{u} \in C^2(D)$ such that

$$-\mathrm{div}\left(\frac{\nabla \tilde{u}(x)}{\sqrt{1+|\nabla \tilde{u}(x)|^2}}\right) = H \quad \text{for all } x \in D.$$

FIRST TRIAL: THE ARGUMENT WITH DG-ST-SI THEOREM

Suppose that: Ω is not a ball, $\mathcal{H}^{d-2}(\partial E \cap \partial\Omega) = 0$, ($\partial\Omega \in C^1$). Pick $x \in \partial E \cap \partial\Omega$. ($d' = d-1$).

a) $\partial E \cap \partial\Omega \cap Q_r(x) := \{(x, f_E(x)),\ x \in \gamma\}$;

b) $\mathcal{H}^{d'-1}(\gamma) \le C\mathcal{H}^{d'-1}(\partial E \cap \partial\Omega \cap Q_r(x)) \le \mathcal{H}^{d-2}(\partial E \cap \partial\Omega) = 0$

$$\begin{cases} -\mathrm{div}\left(\frac{\nabla f_E(x)}{\sqrt{1+|\nabla f_E(x)|^2}}\right) = h(\Omega) \ \text{ on } D \setminus \gamma \\ \mathcal{H}^{d'-1}(\gamma) = 0,\ f_E : \mathbb{R}^{d'} \to \mathbb{R} \end{cases}$$

c) Then DG,St-Si Theorem

$$-\mathrm{div}\left(\frac{\nabla f_E(x)}{\sqrt{1+|\nabla f_E(x)|^2}}\right) = h(\Omega) \ \text{ on } D$$

and thus ∂E has constant mean curvature equal to h.

d) Alexandrov's Theorem (revised): E is a ball and thus Ω is a ball. Contradiction: we assumed $\Omega \neq B$.

$$\textbf{Theorem}: \ \mathcal{H}^{d-2}(\partial E \cap \partial\Omega) > 0.$$

Pokrovskii's Theorem

Let $u \in C^2(D \setminus \gamma) \cap C^{1,\alpha}(D)$, $D \subset \mathbb{R}^{d'}$ satisfies

$$-\text{div}\left(\frac{\nabla u(x)}{\sqrt{1+|\nabla u(x)|^2}}\right) = H \quad \text{for all } x \in D \setminus \gamma$$

and $\mathcal{H}^{d'-1+\alpha}(\gamma) = 0$ then there exists a unique extension $\tilde{u} \in C^2(D)$ such that

$$-\text{div}\left(\frac{\nabla \tilde{u}(x)}{\sqrt{1+|\nabla \tilde{u}(x)|^2}}\right) = H \quad \text{for all } x \in D.$$

Pokrovskii's removability applies to: Constant Mean Curvature equation, p-laplacian equation, and (lately) uniformly elliptic equations in divergence form.[13]

13

- Pokrovskii, *Removable singularities of p-harmonic functions*. Differential Equations, 41(7):941–952, 2005
- Pokrovskii, *Removable singularities of solutions of second-order divergence-form elliptic equations*. Mathematical Notes, 77(3-4):391–399, 2005;
- Pokrovskii, *Removable singularities of solutions of the minimal surface equation*. Functional Analysis and Its Applications, 39(4):296–300, 2005;
- Pokrovskii, *Removable singularities of solutions of elliptic equations*. Journal of Mathematical Sciences, 160(1):61–83,2009.

Pokrovskii's Theorem as a Divergence property

C.C. observation 2020[14]: the structure of quasi-linear elliptic equation is not required. Indeed if $F \in C^{0,\alpha}(D;\mathbb{R}^{d'})$ satisfies

$$\int_D \mathrm{div}(\phi) F \,\mathrm{d}x = \int_D \phi g \,\mathrm{d}x \text{ for all } \phi \in C_c^\infty(D \setminus \gamma), \quad -\mathrm{Div}(F) = g \text{ on } D \setminus \gamma$$

and γ closed set with $\mathcal{H}^{d'-1+\alpha}(\gamma) = 0$ then

$$\int_D \mathrm{div}(\phi) F \,\mathrm{d}x = \int_D \phi g \,\mathrm{d}x \text{ for all } \phi \in C_c^\infty(D), \quad -\mathrm{Div}(F) = g \text{ on } D.$$

[14] Firstly observed, for $\alpha = 0$, in:
Ponce, *Singularities of the divergence of continuous vector fields and uniform hausdorff estimates* Indiana University Mathematics Journal, pages 1055–1074, 2013

SECOND TRIAL: THE ARGUMENT WITH POKROVSKII'S THEOREM

Suppose that: Ω is not a ball, $\mathcal{H}^{d-2+\alpha}(\partial E \cap \partial\Omega) = 0$, ($\partial\Omega \in C^{1,\alpha}$). Pick $x \in \partial E \cap \partial\Omega$ ($d' = d-1$)

a.0) $\partial E \cap \partial\Omega \cap Q_r(x) := \{(x, f_E(x)),\ x \in \gamma\}$;

a.1) If $\partial\Omega \in C^{1,\alpha} \Rightarrow \partial E \in C^{1,\alpha}$ around $x \in \partial E \cap \partial\Omega$[15];

b) $\mathcal{H}^{d'-1+\alpha}(\gamma) \le C\mathcal{H}^{d-2+\alpha}(\partial E \cap \partial\Omega) = 0$ and

$$\begin{cases} -\mathrm{div}\left(\frac{\nabla f_E(x)}{\sqrt{1+|\nabla f_E(x)|^2}}\right) = h(\Omega) \ \text{ on } D \setminus \gamma \\ \mathcal{H}^{d'-1+\alpha}(\gamma) = 0,\ f_E \in C^{1,\alpha}(D) \end{cases}$$

c) Then, Pokrovskii's Theorem

$$-\mathrm{div}\left(\frac{\nabla f_E(x)}{\sqrt{1+|\nabla f_E(x)|^2}}\right) = h(\Omega) \ \text{ on } D$$

and thus ∂E has constant mean curvature equal to h.

d) Alexandrov's Theorem (revised): E is a ball and thus Ω is a ball. Contradiction: we assumed $\Omega \neq B$.

$$\textbf{Theorem}:\ \mathcal{H}^{d-2+\alpha}(\partial E \cap \partial\Omega) > 0.$$

[15] It can be derived as an adaptation of
Giaquinta. *Remarks on the regularity of weak solutions to some variational inequalities*. Mathematische Zeitschrift, 177(1):15–31, 1981.

SHARPNESS IN $d = 2$

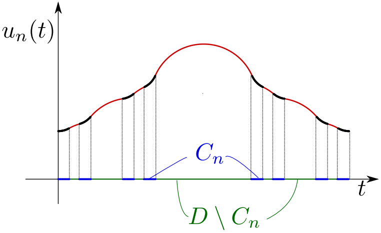
For any $\alpha \in (0,1]$ there exists an open bounded set $\Omega \subset \mathbb{R}^2$ with regularity of class $C^{1,\alpha}$ such that

$$\dim_{\mathcal{H}}(\partial E \cap \partial\Omega) = \alpha \quad (\mathcal{H}^{\alpha}(\partial E \cap \partial\Omega) > 0,\ \mathcal{H}^{s}(\partial E \cap \partial\Omega) = 0 \text{ for } s > \alpha.)$$

How is it built

$D=(0,1)$ and C_n Cantor type construction;

$$u_n(t) := \int_0^t \frac{(s_n(r)-Hr)}{\sqrt{1-(s_n(r)-Hr)^2}}\,\mathrm{d}r,\quad s_n(t) := \frac{1}{\mathcal{L}^2(C_n)}\int_0^t \mathbb{1}_{C_n}(r)\,\mathrm{d}r$$



$$\begin{cases} -\left(\frac{u_n'(t)}{\sqrt{1+|u_n'(t)|^2}}\right)' = H, & \text{on } D\setminus C_n \\ u_n \in C^{1,\alpha}(D)\cap C^\infty(D\setminus C_n) \end{cases}$$

$$\gamma := \bigcap_{n\in\mathbb{N}} C_n,$$

$$u_n \to u$$

$$\dim_{\mathcal{H}}(\gamma) = \alpha,$$

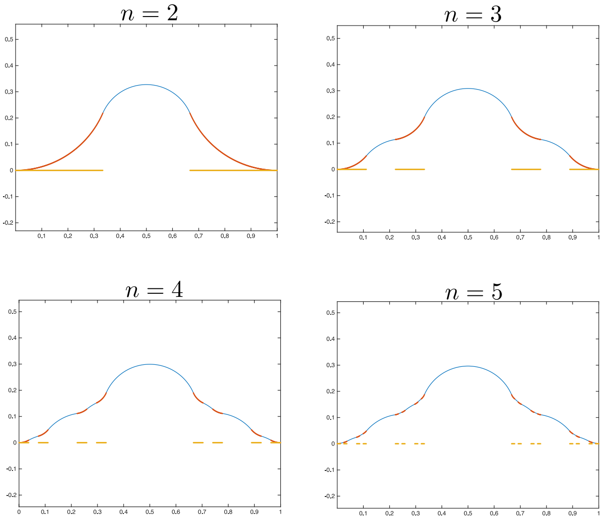
$$\mathcal{H}^\alpha(\gamma) > 0$$

By playing with the construction of C_n any $\alpha \in (0,1)$ can be reached

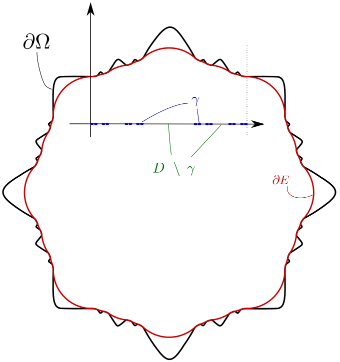
$$\begin{cases} -\left(\frac{u'(t)}{\sqrt{1+|u'(t)|^2}}\right)' = H, & \text{on } D\setminus \gamma \\ u \in C^{1,\alpha}(D)\cap C^\infty(D\setminus \gamma) \end{cases}$$

Moreover: $u_n'(0)=u_n'(1)=0 \Rightarrow u'(0)=u'(1)=0$

$C^{1,\alpha}$ case



$C^{1,\alpha}$ case[16]



We can build Ω around, so that (locally) $\partial E \cap \partial\Omega = (\mathrm{Id}\,, u)(\gamma)$ and

$$h(\Omega) = \frac{P(E)}{\mathcal{L}^2(E)}.$$

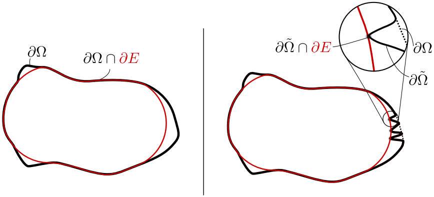
Thence E is a Cheeger set of Ω with contact surface:

$$\dim_{\mathcal{H}}(\partial E \cap \partial\Omega) = \alpha$$

[16] Tools required and other interesting literature about pathological Cheeger sets:

- Leonardi, Neumayer, Saracco. *The cheeger constant of a jordan domain without necks*. Calculus of Variations and Partial Differential Equations, 56(6):164, 2017
- Saracco. *A sufficient criterion to determine planar self-cheeger sets* Journal of Convex Analysis, 28(3), 951-958.
- Leonardi, Saracco,*Two examples of minimal Cheeger sets in the plane*. Annali di Matematica 197, 1511–1531 (2018)

FINAL CONSIDERATION



a) Are the bounds $\mathcal{H}^{d-2+\alpha}(\partial E \cap \partial\Omega) > 0$, sharp also in $d \geq 3$?

(Missing a.1) The couple (u, γ) with the required geometry;

(Missing a.2) Instruments, like the planar one, to build the ambient space Ω;[17]

b) The argument is sensible to the regularity of ∂E more than to the regularity of $\partial\Omega$. That is why, for Ω convex set we can infer $\mathcal{H}^{d-1}(\partial E \cap \partial\Omega) > 0$.

c) Given a convex set Ω, are the bounds true also locally? We expect that either $\mathcal{H}^{d-1}(\partial E \cap \partial\Omega \cap A) > 0$ or $\partial E \cap \partial\Omega \cap A = \emptyset$

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[17] More on these topics:

- Leonardi, Saracco. *Minimizers of the prescribed curvature functional in a Jordan domain with no necks* ." ESAIM: Control, Optimisation and Calculus of Variations 26 (2020): 76;
- Leonardi, Saracco. *The prescribed mean curvature equation in weakly regular domains*. Nonlinear Differential Equations and Applications NoDEA 25.2 (2018): 9.

A (very) partial list of literature on Cheeger sets

1968 *The relation between the laplacian and the diameter for manifolds of non-negative curvature.* J. Cheeger. (Archiv der Mathematik);

1969 *A lower bound for the smallest eigenvalue of the laplacian.* J. Cheeger. (Proceedings of the Princeton conference in honor of Professor S.Bochner);

2007 *On the selection of maximal cheeger sets.* G. Buttazzo, G. Carlier, and M. Comte. (Differential and Integral Equations);

2007 *Uniqueness of the cheeger set of a convex body.* V. Caselles, A. Chambolle, and M. Novaga. (Pacific J. Math.);

2009 *A note on Cheeger sets.* A. Figalli, F. Maggi, A. Pratelli. (Proceedings of the American Mathematical Society);

2010 *Some remarks on uniqueness and regularity of cheeger sets.* V. Caselles, A. Chambolle, and M. Novaga. (Rend. Semin. Mat. Univ. Padova);

2011 *An introduction to the cheeger problem.* E. Parini. (Surv.Math.Appl.);

2015 *An overview on the Cheeger problem.* G. P. Leonardi. (In New trends in shape optimization, Springer);

2016 *A faber–krahn inequality for the cheeger constant of n-gons*, D. Bucur, I. Fragalà. (The Journal of Geometric Analysis);

2016 *On the cheeger sets in strips and non-convex domains.* G. P. Leonardi and A. Pratelli. (Calculus of Variations and Partial Differential Equations) ;

2017 *The cheeger constant of a jordan domain without necks.* G. P. Leonardi, R. Neumayer, and G. Saracco. (Calculus of Variations and Partial Differential Equations);

2018 *On the honeycomb conjecture for a class of minimal convex partitions.* D. Bucur, I. Fragalà, B. Velichkov, and G. Verzini. (Transactions of the American Mathematical Society);

2018 *The prescribed mean curvature equation in weakly regular domains.* G. P. Leonardi and G. Saracco. (Non linear Differential Equations and Applications NoDEA) ;

2018 *Two examples of minimal cheeger sets in the plane.* G. P.Leonardi and G. Saracco. (Annali di Matematica Pura ed Applicata);

Thank you for your attention