Stability of the ball under volume preserving fractional mean curvature flow

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Fractional mean curvature

Joint work with Matteo Novaga.

Let $s \in (0,1)$; $E \subseteq \mathbb{R}^n$, the fractional perimeter is

$$\mathrm{Per}_s(E) = \int_E \int_{\mathbb{R}^n \setminus E} \frac{1}{|x-y|^{n+s}} = \frac{1}{s(n+s-2)} \int_{\partial E} \int_{\partial E} \frac{\nu(x)\cdot\nu(y)}{|x-y|^{n+s-2}}$$

weighted measure of the variation of the exterior normal ν around the bdry
fractional Gagliardo seminorm of χ_E
It is an interpolation norm between the L^1 norm and the BV norm

The fractional mean curvature is the first variation of Per_s, and if $E \in C^{1,\alpha}$ for some $\alpha > s$, then

$$H_E^s(x) = \int_{\mathbb{R}^n} \frac{\chi_{E^c}(y) - \chi_E(y)}{|x-y|^{n+s}} dy = \frac{2}{s} \int_{\partial E} \frac{(y-x)\cdot\nu(y)}{|x-y|^{n+s}} dH^{n-1}(y).$$

$\lim_{s\to 1}(1-s)H_E^s(x) = cH_E(x)$ and $\lim_{s\to 0} sH_E^s(x) = c\omega_n$.

Sets having constant/ almost constant fractional curvature

- Let $E \subseteq \mathbb{R}^n$ $C^{1,s}$ bdd open set, with $H^s_E(x) \equiv c$, then $E = B$
  regular critical pts of the isoperimetric pbm are single spheres
- Let $E \subseteq \mathbb{R}^n$ $C^{2,s+\varepsilon}$ bdd open set, define

$$\eta_s(E) = \frac{\operatorname{diam}(E)^{s+n+1}}{|E|^2} \sup_{x,y \in \partial E} \frac{|H^s_E(x) - H^s_E(y)|}{|x-y|}.$$

  There exists $C_{n,s} > 0$ such that if $\eta_s(E) < C_{n,s}$ then
  ∂E is a $C^{2,s-\alpha}$ graph on the sphere, for all $\alpha > 0$, with norm linearly bounded by $\eta_s(E)$.

Ref: **Cabré, Fall, Sola-Morales, T. Weth** [2018]
Ciraulo, Figalli, Maggi, Novaga [2018]
Methods: Moving planes, strong maximum principle/ regularity theory for nonlocal equations

In the local case: Among sets of finite perimeter and finite volume, finite unions of balls with equal radii are the unique critical points of the Euclidean isoperimetric problem (ref: **Delgadino, Maggi, 2019**, by . Heintze-Karcher inequality, maximum principle and regularity results).

In the fractional setting a similar result is missing.
Up to now: any smooth λ-minimizer of the fractional perimeter is a single sphere
so, the result holds e.g when

either $n = 2$

or $n \leq 7$ and $s \geq 1 - \varepsilon_0$,

any λ-minimizer of the fractional perimeter is a single sphere.
(ref for regularity of local minimizers of the fractional perimeter: **Savin Valdinoci 2013, Valdinoci Caffarelli, 2011**)

Volume preserving fractional mean curvature flow

We introduce the geometric evolution law for $t > 0$:

$$(MCF) \qquad \partial_t x_t \cdot \nu(x_t) = -H^s_{E_t}(x_t) + \lambda(t) \qquad x_t \in \partial E_t.$$

$E_0 \subseteq \mathbb{R}^n$ is the i.d. and λ is a Lagrange multiplier corresponding to the volume constraint $|E_t| = |E_0|$: when E_t is sufficiently regular

$$\lambda(t) = \overline{H^s_{E_t}} = \frac{1}{H^{n-1}(\partial E_t)} \int_{\partial E_t} H^s_{E_t}(y) dH^{n-1}(y).$$

The flow is nonlocal due to the presence of the forcing term $\lambda(t)$.

Gradient flow structure:

$$\frac{d}{dt}\mathrm{Per}_s(E_t) = -\int_{\partial E_t} \left| H^s_{E_t}(x) - \overline{H^s_{E_t}} \right|^2 dH^{n-1}(x).$$

Volume preserving mean curvature flow

In the classical case

$$\partial_t x_t \cdot \nu(x_t) = -H_{E_t}(x_t) + \frac{1}{H^{n-1}(\partial E_t)} \int_{\partial E_t} H_{E_t}(y) dH^{n-1}(y)$$

- Smooth solutions for short times, **Gage 85, Huisken 87**
- Volume preserving version of discrete scheme Luckhaus and Sturzenhecker, **Mugnai, Seis, Spadaro, 2016, Morini, Ponsiglione, Spadaro, 2022**, and flat flow (L^1_{loc} limit) **Julin, Niinikoski, 2020**, **Julin, Morini, Ponsiglione, Spadaro, ppt 2022**. Consistency of the flat flow with smooth slns when i.d. is $C^{1,1}$, recently addresses **Julin, Niinikoski, ppt 2022**.
- varifold solution (Brakke flow), **Takasao 2015, 2022**,
- even if no comparison principle available, a notion of viscosity solutions (with L^1_{loc} forcing), by fixing the Lagrange-multiplier for competitors, **Kim, Kwon, 2020**.
- consistency of weak solns with strong slns recently addressed **Laux, ppt 2022** .

Smooth solutions

Back to the fractional case:

- For compact i.d. $E_0 \in C^{1,1}$, there exists a C^∞ solution for short time
  ref: **Julin, La Manna, 2020**
  Schauder estimates and fixed point argument
- For compact convex i.d. $E_0 \in C^{2,s+\varepsilon}$, under the additional assumption that the solution remains smooth and does not develop singularities as long as the fractional curvature is bounded, the flow exists smooth and convex for all times
  Moreover E_t converges as $t \to +\infty$ to a ball, up to translations, i.e. the limit set, geometrically a sphere, could keep translating indefinitely.
  ref: **Cinti, Sinestrari, Valdinoci, 2020**, estimates on inner/outer radii, a priori weighted estimates on the evolving surface

Minimizing movements and flat flow

A recent work by **De Gennaro, Kubin, Kubin, ppt 2022**: following Luckhaus and Sturzenhecker, Almgrem, Taylor, Wang they define a discrete in time variational approximation scheme of the flow, which incorporates a volume penalization

- the discrete flow is converging exponentially fast to a ball (in the case $n = 2$ or $n \leq 7$ and $s \geq 1 - \varepsilon_0$)
- the fractional flat flow, defined as the L^1_{loc} limit points of the discrete flow, is volume preserving, under an additional assumption of uniform boundedness of the discrete flow

In the local case for the discrete flow ref:**Mugnai, Seis, Spadaro, 2016**, **Morini, Ponsiglione, Spadaro 2022** , for the flat flow, in dim 2,3 **Julin, Niinikoski 2020**, convergence up to translation and without convergence rate, in dim 2 **Julin, Morini, Ponsiglione, Spadaro, ppt 2022** exponential convergence

Nearly spherical sets

We restrict considering as i.d. nearly spherical sets:

$$E := \{rx, x \in \partial B_m, r \in [0, 1 + u(x)]\}$$

where B_m is the ball with volume m,

$$u : \partial B_m \to \mathbb{R} \qquad \text{is a } C^{1,1} \text{ function with } \|u\|_{C^1} < 1.$$

From now on, $m = 1$: all the results can also be stated for sets parametrized over a ball B_m with constants depending on m (and the dfependance on m is uniform for m in bdd intervals).

Alexandrov type estimate for nearly spherical sets

Define the (squared) fractional Gagliardo seminorm of u:

$$[u]^2_{\frac{1+s}{2}} := \int_{\partial B}\int_{\partial B} \frac{(u(x)-u(y))^2}{|x-y|^{n+s}} dH^{n-1}(y)dH^{n-1}(x).$$

The hypersingular Riesz operator on the sphere:
$(-\Delta)^{\frac{1+s}{2}}u(x) := 2\int_{\partial B} \frac{u(x)-u(y)}{|x-y|^{n+s}} dH^{n-1}(y)$. Note that
$[u]^2_{\frac{1+s}{2}} = \int_{\partial B} u(x)(-\Delta)^{\frac{1+s}{2}}u(x)dH^{n-1}(x)$.

Theorem

E is a nearly spherical set with $|E| = |B|$ and $\int_E xdx = 0$. Then $\exists C(n,s) > 0$ and $\varepsilon_0(n,s) \in (0,1)$ such that if $\|u\|_{C^1} < \varepsilon_0$ there holds

$$(A) \qquad [u]^2_{\frac{1+s}{2}} + \|u\|_2^2 \le C(n,s)\|H^s_E - \overline{H^s_E}\|^2_{L^2(\partial E)}.$$

Moreover $\exists K(n,s) > 0$ such that

$$(L) \qquad \mathrm{Per}_s(E) - \mathrm{Per}_s(B) \le K(n,s)\|H^s_E - \overline{H^s_E}\|^2_{L^2(\partial E)}.$$

This provides a provides a linear control on the $H^{\frac{1+s}{2}}$ distance from the reference sphere of a nearly spherical set, in terms of the L^2 deficit of the fractional curvature.

an analogous estimate has been proved by slighlty different methods in **De Gennaro, Kubin, Kubin, ppt 2022**, also taking into account the stability of the estimate as $s \to 1$.

In the local setting: **Krummel, Maggi, 2017**, **Morini, Ponsiglione Spadaro, 2022**.

Proof

Proof of (A):

- integral estimates on (the Taylor approximations of) $H^s_E(x) - H^s_B$ and $u(x)(H^s_E(x) - H^s_B)$, in term of the $H^{\frac{1+s}{2}}$ norm of u
- write the Fourier serie of u in terms of the spherical harmonics Y^j_i (up to degree 2) and use a fractional Poincaré type inequality (for $u - aY_0 - \sum_i b_i \cdot Y^i_1$)
- obtain an estimate of the $H^{\frac{1+s}{2}}$ norm in terms of $\|H^s_E - H^s_B\|^2_{L^2(\partial B)}$ and conclude by a rescaling argument.

Proof of (L):

- by (A) using Fuglede type estimates:

$$k(n)([u]^2_{\frac{1+s}{2}} + s\mathrm{Per}_s(B)\|u\|^2_2) \leq \mathrm{Per}_s(E) - \mathrm{Per}_s(B) \leq c(n)[u]^2_{\frac{1+s}{2}}.$$

ref:**Figalli, Fusco, Maggi, Millot, Morini, 2015**
Di Castro, Novaga, Ruffini, Valdinoci, 2015.

Smooth flow and exponential convergence

Theorem

Let E_0 be a nearly spherical set on a given ball B, with $|E_0| = |B|$. For any $C > 0$ there exists $\varepsilon = \varepsilon(C) > 0$ such that if $\|u_0\|_{C^{1,1}} \leq C$ and $\|u_0\|_{C^1} < \varepsilon$, then the flow E_t starting from E_0 exists smooth for every time t and $E_t - \bar{b} \to B$ in C^∞, for some $\bar{b} \in \mathbb{R}^n$. Moreover, there exists a constant $C(n,s)$ depending on n, s such that

$$\mathrm{Per}_s(E_t) - \mathrm{Per}_s(B) \leq C(n,s)(\mathrm{Per}_s(E_0) - \mathrm{Per}_s(B))e^{-C(n,s)t} \qquad \forall t \geq 0$$

and for all $m \geq 1$ there exists a constant $C(m,n,s) > 0$

$$\|u(x,t) - (\bar{b}\cdot x)x\|_{C^m(\partial B)} \leq C(m,n,s)(\mathrm{Per}_s(E_0) - \mathrm{Per}_s(B))e^{-C(m,n,s)t} \qquad \forall t \geq$$

Exponential convergence of convex sets

As a corollary we get an improvement of the result by Cinti, Sinestrari, Valdinoci
ruling out the the possibility of indefinite translations and giving the exponential rate of convergence.

Analogous results in the local setting: **Huisken 1987** for convex sets, **Escher, Simonett, 1998**, and **Antonopoulou, Karali, Sigal, 2010** for nearly spherical sets.

Lojasiewicz-Simon inequality

(L) can be interpreted as a Lojasiewicz-Simon inequality for Per_s:

$$(\mathrm{Per}_s(E) - \mathrm{Per}_s(B))^{\frac{1}{2}} \leq C(n,s)\|H_E^s - \overline{H_E^s}\|_{L^2}$$

- **Lojasiewicz**: in the 50s gives an upper bound for the distance from a point to the nearest zero of a real analytic function: $p \in Z$ $\exists U$ and $\alpha \in [1/2, 1)$ s.t. $|f(x) - f(p)|^\alpha \leq C|\nabla f(x)|$
- in the 80s, by **Simon**: infinite dimensional version (and from then many extensions, **Rupp, 2020** submanifolds) with applications to uniqueness of limits for gradient flows and also singularities of nonlinear PDE's (ref: **Pozzetta, Pozzetta Pluda ppt 2022**, elastic flow, minimal networks, **Colding Minicozzi, 2015**, uniqueness of blow up..)

Brief sketch:

- for short time the flow exists smooth and the set remains nearly spherical, let $u(t,\cdot)$ the height function
- $\|u_t\|_{L^2} \sim C\|H^s_{E_t} - \overline{H^s_{E_t}}\|_{L^2} \overset{L}{\leq} \frac{\|H^s_{E_t}-\overline{H^s_{E_t}}\|^2_{L^2}}{((\mathrm{Per}_s(E)-\mathrm{Per}_s(B))^{\frac{1}{2}}} \sim -\frac{d}{dt}\left(\mathrm{Per}_s(E) - \mathrm{Per}_s(B)\right)^{\frac{1}{2}}$
- first of all $\|u(t,\cdot)\|_{L^2}$ remains small, and by interpolation inequalities we get that also $\|u(t,\cdot)\|_{C^{1+s+\alpha}}$ remains small, and the flow can be continued for all times
- in particular the control in $L^1(t,+\infty)$ of $\|u_t(t,\cdot)\|_{L^2}$ give that $\|u(t,\cdot)\|_{L^2}$ is Cauchy and again by interpolation $\|u(\cdot,t_2) - u(\cdot,t_1)\|_{C^m(\partial B)} \leq C(m,n,s)e^{-C(m,n,s)t_1}$.

Open problems: sharp quantitative versions of the Alexandrov's theorem

- for the local perimeter: there exists ε s.t. if $E \subseteq \mathbb{R}^2$ is a bdd C^2 set with $\mathrm{Per}(E) \leq M$, $|E| = m$ and $\|H_E - \bar{H}_E\|_2 \leq \varepsilon$ then E is C^1 diffeomorphic to a union of N disjoint disks of equal radii d

$$|\mathrm{Per}(E) - N2\pi d| \leq C_M \|H_E - \bar{H}_E\|_2^2$$

  and very connected component is a nearly spherical set over one of these disks
  Julin, Morini, Ponsiglione, Spadaro, ppt 2022, previously **Fusco, Julin, Morini, 2022** with L^1 norm of the curvature deficit.

Is it true also in the fractional case?

- A related problem: compactness results for uniformly bounded sets with vanishing L^2 deficit (see for the local case **Delgadino, Maggi, Mihaila, Neumayer, 2018**).
- Inequality (L) holds true for general bounded open sets $E \in C^{1,1}$?

$$\mathrm{Per}_s(E) - \mathrm{Per}_s(B) \leq K(n,s) \| H_E^s - \overline{H_E^s} \|^2_{L^2(\partial E)}$$

In the local case,in dim 2, by **Julin, Morini, Ponsiglione, Spadaro,ppt 2022** used to prove exponential convergence of the flat flow to a disjoint union of balls with equal size.

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Thanks for the attention