Isoperimetric Problems

Report of Contributions

https://events.dm.unipi.it/e/isoperimetric

Contribution ID: **77** Type: **not specified**

On a quantitative isoperimetric inequality involving the barycentric asymmetry

Monday, 20 June 2022 09:30 (50 minutes)

In 1993 Fuglede showed a sharp quantitative isoperimetric inequality, namely, that for convex sets in R^N the isoperimetric deficit controls the square of the barycentric asymmetry. The convexity assumption is used because the result is clearly false for general sets. Estimates of this form have then then studied in the following years, for instance by Cicalese-Leonardi, and by Bianchini-Croce-Henrot, who also consider the connectedness assumption in the plane. We will discuss the general question, reviewing the main ideas, and we will present a very recent result in this direction.

Presenter: PRATELLI, Aldo (Università di Pisa)

Contribution ID: **79** Type: **not specified**

Steiner Problem, global and local minimizers of the Length functional

Monday, 20 June 2022 10:20 (50 minutes)

The Steiner problem, in its classical formulation, is to find the 1-dimensional connected set in the plane with minimal length that contains a finite collection of points.
Although existence and regularity of minimizers is well known, in general finding explicitly a solution is extremely challenging, even numerically.
A possible tool to validate the minimality of a certain candidate is the notion of calibrations.
In this talk I will introduce the different definitions of calibrations for the Steiner problem available in the literature,
I will give example of existence and non—existence of calibrations and I will show how one can easily get informations on both global and local minimizers.

Presenter: PLUDA, Alessandra (Università di Pisa)

Contribution ID: **81** Type: **not specified**

Isoperimetric problems on periodic lattices

Monday, 20 June 2022 15:00 (50 minutes)

Motivated by the crystallization issue, we focus on the minimization of Heitman-Radin potential energies for configurations of N particles in a periodic lattice, and in particular on the connection with anisotropic isoperimetric problems in the suitably rescaled limit as $N \to \infty$. Besides identifying the asymptotic Wulff shapes through Gamma-convergence, we obtain fluctuation estimates for quasiminimizers that include the well-known $N^{3/4}$ conjecture for minimizers in planar lattices. Our technique combines the sharp quantitative Wulff inequality with a notion of quantitative closeness between discrete and continuum problems. These results have been obtained in collaborations with Marco Cicalese and Leonard Kreutz.

Presenter: LEONARDI, Gian Paolo (Università di Trento)

Contribution ID: **82** Type: **not specified**

An isoperimetric problem with strong capacitary repulsion

Monday, 20 June 2022 15:50 (50 minutes)

In a project started 8 years ago, together with M. Goldman, C. B. Muratov and M. Novaga, we studied a variational problem aiming to describe the behavior of a liquid electrically charged drop. This leads to an interesting isoperimetric problem in presence of a capacitary repulsive term. Tuning on the parameters of the energy, such as the amount of the charge, the volume of the drop and the choice of the Riesz potential inherent in the definition of capacity, one observes different behaviors. In particular the problem becomes quite non-trivial in a certain class of singular capacities. In this talk, after a general overview of the topic, I will concentrate on a solution we recently obtained for this latter issue.

Presenter: RUFFINI, Berardo

Contribution ID: **85** Type: **not specified**

Lecture 1. Introduction to isoperimetry in Riemannian manifolds and the emergence of nonsmooth structures

Tuesday, 21 June 2022 09:30 (50 minutes)

In this introductory talk, I will recall the main issues in dealing with the isoperimetric problem on Riemannian manifolds.
I will then focus on manifolds with lower bounds on the Ricci curvature, and discuss some fundamental tools in this setting.
Their rigorous derivation and their applications will naturally call for the analysis on nonsmooth RCD structures, and will be the object of the subsequent lectures.

Presenter: FOGAGNOLO, Mattia (Centro De Giorgi Pisa)

Contribution ID: **86** Type: **not specified**

Lecture 2. Generalized existence and compactness for isoperimetric regions and applications

Tuesday, 21 June 2022 10:20 (50 minutes)

In this talk, we want to give an overview of a result of generalized existence and compactness of isoperimetric regions in the context of smooth (possibly noncompact) Riemannian manifolds without boundaries and of bounded geometry together with metric theoretic proofs that for almost-isoperimetric regions small volumes implies small diameters always in the context of smooth Riemannian manifolds. Applications to the theory of the existence of isoperimetric regions in noncompact Riemannian manifolds, to the Aubin-Cartan-Hadamard conjecture for small volumes, and the estimates of a topological lower bound for the number of solutions of the Cahn-Hilliard equation and Cahn-Hilliard systems are given (provided time permits).

Presenter: NARDULLI, Stefano (Universidade Federal do ABC)

Contribution ID: **88** Type: **not specified**

Quantitative stability estimates for some fractional inequalities

Tuesday, 21 June 2022 11:40 (50 minutes)

We present stability results for some functional inequalities (such as the Faber-Krahn and the isocapacitary inequality) in the nonlocal setting. The proof is based on some ideas by Hansen and Nadirashvili (who considered the classical local case) and uses the so-called Caffarelli-Silvestre extension.

Presenter: CINTI, Eleonora (Università di Bologna)

Contribution ID: **90** Type: **not specified**

Classification of C^2 solutions for the Finsler isoperimetric problem in H^1

Tuesday, 21 June 2022 15:00 (50 minutes)

In the Heisenberg group $H^1 = \mathbb{R}^3$ we consider the
perimeter associated with a norm on the horizontal distribution.
The existence of isoperimetric sets is well-known. Assuming the C^2_+ regularity of the norm, we are able to classify isoperimetric sets of class C^2.
This is an extension to the Finsler case of a result by Ritorè and Rosales. Isoperimetric sets turn out to be foliated by geodesics for a natural optimal control problem. This property is consistent with Pansu's conjecture.
This is a joint work with Franceschi, Righini and Sigalotti.

Presenter: MONTI, Roberto (Università di Padova)

Contribution ID: **91** Type: **not specified**

Minimal clusters in the plane with double densities

Monday, 20 June 2022 11:40 (50 minutes)

The aim of this seminar is to present some results about the isoperimetric problem for clusters in the plane with double density. This amounts to finding the best configuration of m regions in the plane enclosing given volumes, in order to minimize their total perimeter, in the case where volume and perimeter are weighted by suitable densities.
We focus on the so-called "Steiner" property, ensuring that boundaries of minimal clusters are made of regular curves meeting in triple points. In the standard Euclidean case the directions at triple points are at 120 degrees. We show that the Steiner property can be generalized to a wide class of densities under natural assumptions. If the perimeter density is isotropic, i.e., it does not depend on the normal, we show that triple points enjoy the usual 120 degrees Steiner property. For anisotropic densities the situation is more delicate and we discuss what possible directions occur in minimizers. Examples will be also discussed.
This is a joint collaboration with A. Pratelli and G. Stefani.

Presenter: FRANCESCHI, Valentina (Università di Padova)

Contribution ID: **94** Type: **not specified**

Lecture 3. Topological properties of isoperimetric sets on RCD spaces

Wednesday, 22 June 2022 09:30 (50 minutes)

In the first part of the lecture, I provide an overview on the differential calculus in the setting of RCD spaces, with a particular focus on the theory of sets of finite perimeter. In the second part of the lecture, I show how to apply this differential calculus in order to obtain a Deformation Lemma, which in turn can be used (under suitable non-collapsing assumptions) to prove that the essential interior of an isoperimetric set is an open bounded set having Ahlfors regular topological boundary.

Presenter: PASQUALETTO, Enrico (SNS Pisa)

Contribution ID: **95** Type: **not specified**

Lecture 4. First and second variation of the area on RCD spaces

Wednesday, 22 June 2022 10:20 (50 minutes)

The first and second variation of the area are cornerstones in classical Geometric Measure Theory and they are at the heart of its connections with Ricci curvature. In this lecture I will illustrate how they can be estimated for perimeter minimizers and isoperimetric sets on RCD spaces, avoiding any regularity theory.

Presenter: SEMOLA, Daniele (Oxford University)

Contribution ID: **97** Type: **not specified**

On compactness and generalized existence results for clusters in Riemannian manifolds with bounded geometry

Wednesday, 22 June 2022 11:40 (50 minutes)

In this talk, we will show a generalized compactness theorem for sequences of clusters with uniformly bounded perimeter and volume in a Riemannian manifold with bounded geometry. The arguments presented in the proof of this generalized compactness theorem when applied to minimizing sequences of clusters give a generalized existence theorem for isoperimetric clusters. To achieve this goal, we show that isoperimetric clusters are bounded and also we prove the local Holder continuity of the multi-isoperimetric profi
le. This work generalizes to the context of Riemannian isoperimetric clusters some previous results about the classical Riemannian isoperimetric problem as well as results from clusters theory in the Euclidean setting.

Presenter: RESENDE DE OLIVEIRA, Reinaldo (Universidade de São Paulo)

Contribution ID: **98** Type: **not specified**

Contact surface of Cheeger sets

Tuesday, 21 June 2022 15:50 (50 minutes)

Geometrical properties of Cheeger sets have been deeply studied by many authors since their introduction, as a way of bounding from below the first Dirichlet (p)-Laplacian eigenvualue. They represent, in some sense, the first eigenfunction of the Dirichlet (1)-Laplacian of a domain. In this talk we will introduce a recent property, studied in collaboration with Simone Ciani, concerning their contact surface with the ambient space. In particular we will show that the contact surface cannot be too small, with a lower bound on the (Hasudorff) dimension strictly related to the regularity of the ambient space. The talk will focus on the introduction of the problem and on the proof of the dimensional bounds. Functional to the whole argument is the notion of removable singularity, as a tool for extending solutions of pdes under some regularity constraint. Finally examples providing the sharpness of the bounds in the planar case are briefly treated.

Presenter: CAROCCIA, Marco (Politecnico di Milano)

Contribution ID: **99** Type: **not specified**

Lecture 5. Sharp differential inequality for the isoperimetric profile on noncollapsed spaces with lower Ricci bounds, and sharp and rigid isoperimetric inequality for nonnegatively curved spaces

Thursday, 23 June 2022 09:30 (50 minutes)

In this talk we will explore two consequences of the results explained in the previous lectures. First, we shall prove a sharp differential inequality for the isoperimetric profile of N-dimensional RCD(K,N) spaces with uniform lower bounds on the volume of unit balls. This inequality is new even in the smooth noncompact setting. We will discuss some consequences of this inequality. Second, we shall give a new proof of the sharp isoperimetric inequality in the setting of N-dimensional RCD(0,N) spaces with Euclidean volume growth. We show that the equality case is reached precisely by balls around the tips of N-dimensional Euclidean metric measure cones. For the rigidity part we ask no regularity on the boundary, and this is new even in the smooth case.

Presenter: ANTONELLI, Gioacchino (SNS Pisa)

Contribution ID: **100** Type: **not specified**

Lecture 6. Isoperimetric sets of large volume on spaces with nonnegative Ricci curvature and Euclidean volume growth

Thursday, 23 June 2022 10:20 (50 minutes)

I will present a new existence result for isoperimetric sets of large volume on manifolds with nonnegative Ricci curvature and Euclidean volume growth, under an additional assumption on the structure of tangent cones at infinity. After a brief discussion on the sharpness of the additional assumption, I will show that it is always verified on manifolds with nonnegative sectional curvature. I will finally present the main ingredients of proof emphasizing the key role of nonsmooth techniques presented in the previous lectures.

Presenter: BRUÈ, Elia (IAS Princeton)

Contribution ID: **102** Type: **not specified**

Stability of the ball under volume preserving fractional mean curvature flow

Thursday, 23 June 2022 11:40 (50 minutes)

I will consider the volume preserving fractional mean curvature flow of a nearly spherical set, showing long time existence and exponential convergence to a ball. The main technical tool used in the proof is a quantitative Alexandrov type estimate for nearly spherical sets.
The result applies in particular to convex initial data under the assumption of global existence.

Presenter: CESARONI, Annalisa (Università di Padova)

Contribution ID: **104** Type: **not specified**

Capillary surfaces and a model of nanowire growth

Thursday, 23 June 2022 15:00 (50 minutes)

After recalling the classical variational formulation of the capillarity problem and some related results, we consider a model for vapor-liquid-solid growth of nanowires proposed in the physical literature. In this model, liquid drops are described as local or global volume-constrained minimizers of the capillarity energy outside a semi-infinite convex obstacle modeling the nanowire. We first address the existence of global minimizers and then, in the case of rotationally symmetric nanowires, we investigate how the presence of a sharp edge affects the shape of local minimizers and the validity of Young's law. Finally, we study the regularity of the contact line between the drop and the nanowire near the sharp edge.

Presenter: MORINI, Massimiliano (Università di Parma)

Contribution ID: **105** Type: **not specified**

Optimal transport and quantitative geometric inequalities

Thursday, 23 June 2022 15:50 (50 minutes)

The goal of the talk is to discuss a quantitative version of the Levy-Gromov isoperimetric inequality (joint with Cavalletti and Maggi) as well as a quantitative form of Obata's rigidity theorem (joint with Cavalletti and Semola). Given a closed Riemannian manifold with strictly positive Ricci tensor, one estimates the measure of the symmetric difference of a set with a metric ball with the deficit in the Levy-Gromov inequality. The results are obtained via a quantitative analysis based on the localisation method via L^1-optimal transport.

Presenter: MONDINO, Andrea (University of Oxford)

Contribution ID: **107** Type: **not specified**

Isoperimetric Multi-Bubble Problems - Old and New

Thursday, 23 June 2022 17:00 (50 minutes)

The classical isoperimetric inequality in Euclidean space $\mathbb{R}^n$ states that among all sets ("bubbles") of prescribed volume, the Euclidean ball minimizes surface area. One may similarly consider isoperimetric problems for more general metric-measure spaces, such as on the sphere $\mathbb{S}^n$ and on Gauss space $\mathbb{G}^n$. Furthermore, one may consider the "multi-bubble" partitioning problem, where one partitions the space into $q \geq 2$ (possibly disconnected) bubbles, so that their total common surface-area is minimal. The classical case, referred to as the single-bubble isoperimetric problem, corresponds to $q = 2$; the case $q = 3$ is called the double-bubble problem, and so on.

In 2000, Hutchings, Morgan, Ritoré and Ros resolved the Double-Bubble conjecture in Euclidean space $\mathbb{R}^3$ (and this was subsequently resolved in $\mathbb{R}^n$ as well) – the optimal partition into two bubbles of prescribed finite volumes (and an exterior unbounded third bubble) which minimizes the total surface-area is given by three spherical caps, meeting at 120°-degree angles. A more general conjecture of J.~Sullivan from the 1990's asserts that when $q \leq n + 2$, the optimal multi-bubble partition of $\mathbb{R}^n$ (as well as $\mathbb{S}^n$) is obtained by taking the Voronoi cells of q equidistant points in $\mathbb{S}^n$ and applying appropriate stereographic projections to $\mathbb{R}^n$ (and backwards).

In 2018, together with Joe Neeman, we resolved the analogous multi-bubble conjecture on the optimal partition of Gauss space $\mathbb{G}^n$ into $q \leq n+1$ bubbles – the unique optimal partition is given by the Voronoi cells of (appropriately translated) q equidistant points. In this talk, we will describe our approach in that work, as well as recent progress on the multi-bubble problem on $\mathbb{R}^n$ and $\mathbb{S}^n$. In particular, we show that minimizing partitions are always spherical when $q \leq n + 1$, and we resolve the latter conjectures when in addition $q \leq 6$ (e.g. the triple-bubble conjecture in $\mathbb{R}^3$ and $\mathbb{S}^3$, and the quadruple-bubble conjecture in $\mathbb{R}^4$ and $\mathbb{S}^4$).

Based on joint work (in progress) with Joe Neeman

Presenter: MILMAN, Emanuel (Technion I.I.T. Haifa)

Contribution ID: **108** Type: **not specified**

Isoperimetric inequalities and potential theory

Friday, 24 June 2022 09:30 (50 minutes)

We describe through some selected examples an approach based on potential theory toward the proof of relevant geometric inequalities, holding both in classical and curved frameworks. Time permitting, we also discuss some applications of interest in general relativity, including the positive mass theorem and the Riemannian Penrose inequality, which - according to G. Gibbons - can be understood as an isoperimetric inequality for black holes.

Presenter: MAZZIERI, Lorenzo (Università di Trento)

Contribution ID: **109** Type: **not specified**

The elastica functional as the critical Gamma limit of a nonlocal isoperimetric problem

Friday, 24 June 2022 10:20 (50 minutes)

I will consider the large mass limit of a nonlocal isoperimetric problem in two dimensions with screened Coulomb repulsion, so that to leading order the nonlocal interaction localizes on the boundary of the sets. For an appropriate choice of screening constant, the perimeter is exactly cancelled out, requiring an analysis of the next order contribution. It turns out that then the nature of the problem changes from length minimization to curvature minimization: I will prove that the Gamma limit is given by (the relaxation of) the elastica functional, i.e., the integral over the squared curvature over the boundary.

This is joint work with Cyrill Muratov and Matteo Novaga.

Presenter: SIMON, Theresa (Universität Münster)

Contribution ID: **114** Type: **not specified**

The charged liquid drop: a dynamic approach

Tuesday, 21 June 2022 17:00 (50 minutes)

We study the motion of charged liquid drop in three dimensions where the equations of motions are given by the Euler equations with free boundary with an electric field. This is a well-known problem in physics going back to the famous work by Rayleigh. Due to experiments and numerical simulations one may expect the charged drop to form conical singularities called Taylor cones, which we interpret as singularities of the flow. In this paper, we study the well-posedness and regularity of the solution. Our main theorem is in the spirit of the Beale-Kato-Majda criterion and roughly states that if the flow remains $C^{1,\alpha}$-regular shape and the velocity remains Lipschitz-continuous, then the flow remains smooth.
This is a joint work with Vesa Julin.

Presenter: LA MANNA, Domenico Angelo (Università di Napoli Federico II)

Contribution ID: **115** Type: **not specified**

Estimates on the Cheeger constant

Monday, 20 June 2022 17:00 (50 minutes)

Given a set $\Omega \subset \mathbb{R}^N$, the Cheeger constant is a purely geometrical quantity defined as the infimum $\mathrm{h}(\Omega) := \inf\left\{ \frac{P(E)}{|E|} \,:\, E \subset \Omega,\, |E| > 0 \right\}$.

Despite seeming unassuming, it pops up in many contexts that apparently have nothing in common. To name a few, under some mild regularity assumptions on Ω: bounds on the first Dirichlet eigenvalue of the p-Laplacian; existence of sets in Ω or of graphs over Ω with prescribed curvature; threshold of vertical load that a flat membrane can sustain before breaking; image reconstruction and denoising. The constant of the unit square has even been a tool in a elementary proof of the Prime Number Theorem!

Given the numerous applications, it is important being able to explicitly compute the constant. This is in general a hard task: a telltale sign is that we do not know the exact value of the constant of the unit cube in dimension $N \geq 3$. The computation is (theoretically) feasible for a large class of Jordan domains in the plane [LNS] or in very special cases in general dimension.

If unable to compute the constant, it would be at least desirable to obtain bounds on it: in [LNS] we proved bounds via interior approximations of the set for 2d domains on which, at least on a theoretical level, the constant can be found by solving an algebraic equation; in [JS] a quantitative inequality for the Cheeger constant has been proved in terms of the Riesz asymmetry; in [BPS] bounds of the constant for cylindrical domains $\Omega = \omega \times [0, L]$ have been shown in terms of the constant of the cross-section ω.

[BPS] G. Buttazzo, A. Pratelli, and G. Saracco. Upper and lower bounds on the first Dirichlet eigenvalue of the p-Laplacian in cylindrical domains, and existence of minimizers of a shape optimization problem. Forthcoming.

[JS] V. Julin and G. Saracco. "Quantitative lower bounds to the Euclidean and the Gaussian Cheeger constants."In: Ann. Fenn. Math. 46.2 (2021), pp. 1071–1087.

[LNS] G. P. Leonardi, R. Neumayer, and G. Saracco. "The Cheeger constant of a Jordan domain without necks."In: Calc. Var. Partial Differential Equations 56.6 (2017), p. 164.

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Presenter: SARACCO, Giorgio (Università di Trento)