Numbers and Numerosities

Dedicated to Mauro Di Nasso in occasione of his 60° birthday

Vieri Benci
Università di Pisa

July 9, 2023

Mille auguri!!!!

The story of Mauro and the Numerosity Theory

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Numerosities of labelled sets: a new way of counting

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Abstract

The notions of "labelled set" and "numerosity" are introduced to generalize the counting process of finite sets. The resulting numbers, called numerosities, are then used to develop nonstandard analysis. The existence of a numerosity function is equivalent to the existence of a selective ultrafilter, hence it is independent of the axioms of ZFC.
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MSC: 03E65; 26E35; 03C20

Keywords: Other hypotheses and axioms; Nonstandard analysis; Ultraproducts and related constructions

0. Introduction

Similarly as cardinals and ordinals, the "numerosities" we present in this paper originate as an attempt to extending the notion of finite cardinality. By considering suitable "labellings", we show that a notion of numerosity for (countable) infinite sets can be defined in such a way that the usual properties of finite cardinalities are preserved. Most notably, the numerosity of a proper subset is strictly smaller than the numerosity of the whole set; the numerosity of a disjoint union is the sum of the numerosities; and the numerosity of a Cartesian product is the product of the numerosities. We remark that these properties together are neither satisfied by cardinal nor by ordinal algebras.

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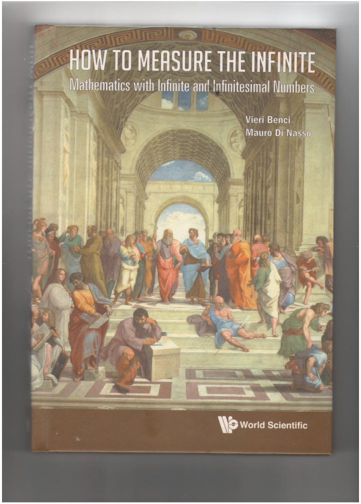
Benci V., Di Nasso M., *How to measure the infinite: Mathematics with infinite and infinitesimal numbers*, World Scientific, Singapore, 2018.



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This talk

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This approach is possible thanks to a new result which I recently obtained.

The comparison relation and the numbers

One of the aims in counting the elements of sets is the comparison of their sizes. We denote by $\preceq$ is a total preorder relation, and as usual, we set

$$X \cong Y :\Leftrightarrow (X \preceq Y \text{ and } Y \preceq X)$$

and

$$X \prec Y :\Leftrightarrow (X \preceq Y \text{ and } Y \ncong X)$$

Definition

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3. Union Principle: If $A \cong A'$ and $B \cong B'$ and $A \cap B = A' \cap B' = \varnothing$, then
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4. Product Principle: If $A \cong A$ and $B \cong B'$ are four families of pairwise disjoint sets, we define the following "product":
$$A \otimes B := \{a \cup b \mid a \in A, b \in B\};$$
then, $$A \otimes B \cong A' \otimes B'.$$

THE NUMBERS

If we have a comparison system then it is possible to build the notion of number:

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Notice that in every set of numbers there are two distingushed elements:

$$0 := \mathfrak{n}(\varnothing)$$

and

$$1 := \mathfrak{n}(\{\varnothing\}).$$

The order relation of numbers

The numbers are a linearly ordered set with the following order relation:

Definition

If $\alpha = \mathfrak{n}(A)$ and $\beta = \mathfrak{n}(B)$

$$\alpha \leq \beta :\Leftrightarrow A \preceq B.$$

The basic operations with numbers

On the set of numbers we can define also the two basic operations: the sum and the product.

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Given two numbers $\alpha = \mathfrak{n}(A)$ and $\beta = \mathfrak{n}(B)$ with $A \cap B = \varnothing$, we set

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Given two numbers $\alpha = \mathfrak{n}(A)$ and $\beta = \mathfrak{n}(B)$ where A and B are as in Def. 1-(4), we set

$$\mathfrak{n}(A) \cdot \mathfrak{n}(B) := \mathfrak{n}(A \otimes B)$$

The basic algebraic properties of numbers

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- existence of the identity elements "0" and "1" with respect to $+$ and $\cdot$
- distributive property

The Cartesian Product

The Cartesian Product plays a prominent role in the development of Mathematics, but probably is not the fundamental notion related to the product between numbers since it is not commutative nor associative namely

$$A \times B \neq B \times A; \; (A \times B) \times C \neq A \times (B \times C) \, .$$

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$$A \times B \neq B \times A; \; (A \times B) \times C \neq A \times (B \times C).$$

However, a good notion of comparison relation must satisfy the following principle:

- Cartesian Product Principle: If A and B are two families of sets as in Def. 1, then

$$A \times B \cong A \otimes B$$

The notion of counting system

In conclusion, we are lead to the following definition:

Definition

Given comparison system $(\mathbb{U}, \preceq)$ which satisfies the Cartesian Product Principle, the corresponding triple $(\mathbb{U}, \mathcal{N}, \mathfrak{n})$ is called **counting system**.

First example

Now let us see some examples of counting systems: we take

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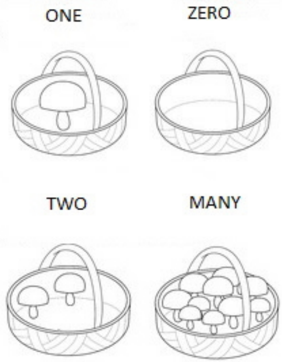
- $\mathbb{U} =$ the class of all sets:
- $\mathcal{N} = \{0, 1, 2, M\}$ where the number M is read **"many".**



First example

Then, there exists a unique comparison relation defined by the following tables:

$+$	$[0]$	$[1]$	$[2]$	$[M]$
$[0]$	0	1	2	M
$[1]$	1	2	M	M
$[2]$	2	M	M	M
$[M]$	M	M	M	M

;

$\cdot$	$[0]$	$[1]$	$[2]$	$[M]$
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$[2]$	0	2	M	M
$[M]$	0	M	M	M

Actually, $(\mathbb{U}, \{0, 1, 2, M\}, \mathfrak{n})$ is not the "smallest" counting system since we can take $(\mathbb{U}, \{0, 1\}, \mathfrak{n})$.

If we want to exclude these interesting, but mathematically trivial examples we need to add some other principle: for example the *V common notion* of Euclid's elements:

The whole is greater than the part.

Euclid's Principle

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The whole is greater than the part.

In our formalism

- **Euclid's principle** - *(V common notion) Given two sets F and G such that F is a proper part of G, then $F \prec G$.*

The natural numbers counting system

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- **Fin** is the class of finite sets
- $\mathbb{N}$ is the set of natural numbers
- $|A| = \mathfrak{n}(A)$ is the number of elements of a finite set.

The Euclid's Principle and infinite sets

If we want to have a universe $\mathbb{U}$ which incudes infinite sets and satisfies the Euclid's Principle we get some limitation.

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For example

Theorem

If $(\mathbb{U}, \mathcal{N}, \mathfrak{n})$ *is a counting system which satisfies Euclid's principle, then* $\mathbb{U}$ *cannot contain sets of rank* ω.

Proof.

If

$$A = \{a, (a, a), (a, a, a), (a, a, a, a), ...\}.$$

then

$$A \times \{a\} = \{(a, a), (a, a, a), (a, a, a, a), ...\} \subset A$$

Proof.

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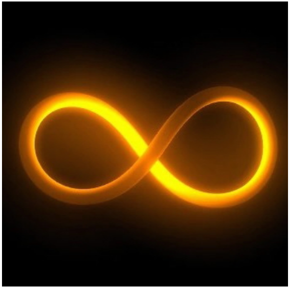
Hence, by Euclid's Principle

$$\mathfrak{n}(A \times \{a\}) < \mathfrak{n}(A)$$

and

$$\mathfrak{n}(A) < \mathfrak{n}(A \times \{a\}) = \mathfrak{n}(A) \cdot \mathfrak{n}(\{a\}) = \mathfrak{n}(A) \cdot 1 = \mathfrak{n}(A).$$

CONTRADICTION! □



So, if we want to have a universe $\mathbb{U}$ which incudes all the infinite sets, the simplest idea is to renounce to Euclid's Principle and to take the counting system $(\mathbb{U}, \mathbb{N} \cup \{\infty\}, \mathfrak{n})$ with the relations

$$\begin{aligned} n + \infty &= \infty, \\ n \cdot \infty &= \infty \text{ for } n \neq 0, \\ 0 \cdot \infty &= 0. \end{aligned}$$

This system, does not have a good algebra: in particular, the equation

$$x + \infty = \infty,$$

has infinitely many solutions, and hence $\mathbb{N} \cup \{\infty\}$ is not even the positive part of an ordered ring.

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In particular, we cannot define infinitesimal numbers such as

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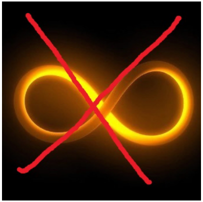
in a consistent way. For this reason, in the History of Mathematics, the symbol "∞" did not even got the dignity of "*number*".



Hume's Principle

Until the XIX century, the idea of natural number was rooted not only on the Euclid's principle, but also on the Hume's Principle

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Until the XIX century, the idea of natural number was rooted not only on the Euclid's principle, but also on the Hume's Principle :

The number of elements in F is equal to the number of elements in G if there is a one-to-one correspondence between F and G.

Hume's Principle

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In our formalism

- **Hume's principle** - *Given two sets F and G, then $F \preceq_{\mathfrak{c}} G$ if there is an injective map*

$$\phi : F \to G.$$

Hume's Principle and the cardinal numbers

It is well known that Euclid's principle and Hume's Principle are satisfied by $(\mathbf{Fin}, \mathbb{N}, |\cdot|)$ but they lead to a contradiction if our universe contains an infinite set.

Hume's Principle and the cardinal numbers

It is well known that Euclid's principle and Hume's Principle are satisfied by $(\mathbf{Fin}, \mathbb{N}, |\cdot|)$ but they lead to a contradiction if our universe contains an infinite set.
Cantor had the great idea to drop Euclid's principle and to use only the preorder relation "$\preceq_{\mathfrak{c}}$" suggested by Hume and introduced the **cardinal numbers**.

The cardinal numbers counting system

The **cardinal numbers counting system** which we will denote by

$$(\mathbb{U},\ \mathbf{Crd},\ |\cdot|).$$

is much reacher than $(\mathbb{U}, \mathbb{N} \cup \{\infty\}, \mathfrak{n})$ since for every set A,

$$|\wp(A)| > |A|$$

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even if the operation tables resemble the trivial one since if β is an infinite number

$$x + \beta = x \cdot \beta = \max\{x, \beta\}$$

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The ordinal numbers do not fit in a counting system in the sense of our definition; however they are related to these systems in several ways. First of all, we can define a triple $(\mathbb{W}, \mathbf{Ord}, \mathfrak{ot})$ similar to a counting system as follows.

- $\mathbb{W}$ is the class of well ordered set;
- **Ord** is the class of ordered numbers,
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- $\forall A \in \mathbb{W}$, $\mathfrak{ot}(A) \in \mathbf{Ord}$ is the order type of A.

$(\mathbb{W}, \mathbf{Ord}, \mathfrak{ot})$ is not a counting system since the operations between ordinals do not satisfy (BAC); we will call it **pseudo-counting system**.

The ordinal numbers

Similarly, we can define a preorder relation on sets in $\mathbb{W}$ as follows:

- $A \preceq_{\mathfrak{o}} B$ if and only there exists a injective map $\Phi : A \longrightarrow B$ which preserves the order, namely $\forall a_1, a_2 \in A$,

$$a_1 <_A a_2 \Rightarrow \Phi(a) <_B \Phi(b).$$

The Numerosity Counting Systems

We have seen that it is not possible to have a counting system which contains infinite sets and which at the same time preserves both Euclid's principle and Hume's principle. However, we can give up Hume's principle and keep Euclid's principle.

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- **Num** is the set of the numerosities;
- for every $A \in \mathbb{U}$, $\mathfrak{num}(A) \in \mathbf{Num}$ is called the numerosity of the set A.

As we have seen, it is not possible to have a numerosity counting system $(\mathbb{U},\ \mathbf{Num},\ \mathfrak{num})$ such that $\mathbb{U}$ is the class of all sets. Thus, it is necessary to restrict $\mathbb{U}$.

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In this case $\mathbb{U}$ is the class of denumerable sets and the counting system takes the form

$$(\mathbb{U},\ \mathbb{N}^*, \mathfrak{num})$$

where $\mathbb{N}^*$ is an elementary extension of the natural numbers.

After this paper the "theory of numerosity" has been developed in several directions and also non-denumerable sets have been included.

The Numerosity Counting Systems

After this paper the "theory of numerosity" has been developed in several directions and also non-denumerable sets have been included.

However no counting system for non-denumerable sets has been described since in (almost) all these models, the Comparison Principle

$$A \preceq B, \textit{ if and only if there exists } A' \subseteq B \textit{ such that } A' \cong A,$$

has not been established.

Thus a question arises naturally:

> *there exists a numerosity counting system in which the sets in* $\mathbb{U}$ *might have any cardinality?*

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The answer is YES and we will discuss this point in the rest of this talk.

Three ways of counting

Before presenting a numerosity counting system in a formal way, I want to give an intuitive idea and to compare it with the cardinal counting system and the ordinal pseudo-counting system

Three ways of counting

In everyday life, there exist (at least) three ways to count the elements of a set:

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3. The third way of counting consists in arranging the elements of a given sets into smaller groups to be counted separately.

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2. In the second way of counting, one arranges the elements of a given set in a row, and then compares such a row with the sequence of natural numbers. This is the concept of number of a five years old child.
3. The third way of counting consists in arranging the elements of a given sets into smaller groups to be counted separately. This is the way of counting of a grown child.

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- Cardinal numbers
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- Numerosities

The third way of counting

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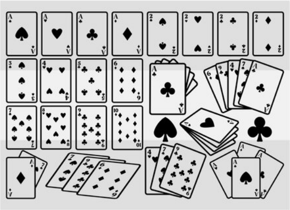

Let us see an example. Assume we are given a big bunch of randomly chosen playing cards. Probably, the better strategy to count the cards is to divide them into smaller decks. For example, one can put all the aces in a deck, all the twos in another deck, and so forth.

The labelled universe

So the third way of counting is possible if the objects of a given set have a "**label**".

The labelled universe

We are lead to a structure formalized by the notion of **labelled universe** that will be described next.

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The formal definition which I will present here has been introduced in

Benci V., Luperi Baglini L., *Euclidean numbers and numerosities,* The Journal of Symbolic Logic, (2022) pp. 1 - 35, DOI: https://doi.org/10.1017/jsl.2022.17.

The labelled universe

From now on we will assume that our universe is given by

$$\Lambda := \{X \in V_\infty(\mathbf{Ato}) \backslash \mathbf{Ato} \mid \ |X| < |\mathbf{Ato}|\},$$

where **Ato** is an infinite set of atoms.

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- $\forall \lambda \in \mathfrak{L}$, $\lambda \cap$ **Fin** $= \varnothing$, namely if $\lambda \in \mathfrak{L}$, λ is a finite set and if $x \in \lambda$, then x is either an atom or an infinite set.

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- the set of labels must be sufficiently large:

$$\bigcup_{\lambda \in \mathfrak{L}} V_\infty(\lambda) = \Lambda \cup \mathbf{Ato},$$

namely $\forall a \in \Lambda \cup \mathbf{Ato},\ \exists\lambda \in \mathfrak{L},\ a \in V_\infty(\lambda)$.

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Example:

$$\mathfrak{L} = \wp_\omega((\Lambda \backslash \textbf{Fin}) \cup \textbf{Ato})$$

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If $a \in V_\infty(\mathbf{Ato})$, we define the label of "a" as follows:

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From now on, $\forall A \in \Lambda \subset V_\infty(\mathbf{Ato})$, we set

$$A_\lambda = \{x \in A \mid \ell(x) \subseteq \lambda\}$$

Properties of the labels

Proposition

The labelling $\ell : V_\infty(\mathbf{Ato}) \to \mathfrak{L}$ *satisfies the following properties:*

1. $\ell(\varnothing) = \varnothing$;
2. $\forall A \in \Lambda$, *the set* A_λ *is finite;*
3. *if* $\{a, b\} \in \Lambda$, $\ell(\{a, b\}) = \ell(a) \cup \ell(b)$;
4. $(A \cup B)_\lambda = A_\lambda \cup B_\lambda$;
5. $(A \otimes B)_\lambda = A_\lambda \otimes B_\lambda$;
6. $(A \times B)_\lambda = A_\lambda \times B_\lambda$;
7. $[\wp(A)]_\lambda = \wp(A_\lambda)$;
8. $(B^A)_\lambda = (B_\lambda)^{A_\lambda}$.

A construction of a numerosity counting system

We take a fine ultrafilter over $\mathfrak{L}$ and we we set:

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Definition

We set

$$A \preceq_{\mathfrak{n}} B$$

if there exists a qualified set $Q \in \mathcal{U}$ such that $\forall \lambda \in Q$

$$|A_\lambda| \leq |B_\lambda| .$$

A construction of a numerosity counting system

It is not difficult to prove that the above definition implies the following fact:

Proposition

$$A \cong_{\mathfrak{n}} B$$

if and only if $\exists$ *a qualified set* Q *and a bijective map* $\Phi : A \to B$ *such that* $\forall \lambda \in Q$,

$$\Phi\left(A_\lambda\right) = B_\lambda$$

The new main theorem

Theorem

The couple $(\Lambda, \preceq_{\mathfrak{n}})$ *is a comparison system that satisfies the Cartesian Product Principle.*

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The couple $(\Lambda, \preceq_{\mathfrak{n}})$ *is a comparison system that satisfies the Cartesian Product Principle.*

Corollary

The numerosity theory $(\Lambda, \mathbf{Num}, \mathfrak{num})$ *is a counting system that satisfies Euclid's Principle.*

Comparison of the numerosities theory with the cardinal and ordinal numbers

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1. The cardinality of A equal to the cardinality of B (in our symbols $A \cong_{\mathfrak{c}} B$) if there exists a bijective map $\Phi : A \to B$.

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$$a \in A, \forall b,\ a \lessdot b \Rightarrow \Phi(a) \lessdot \Phi(b)$$
3. The numerosity of $A \in \Lambda$ is equal to the the numerosity of $B \in \Lambda$ (in our symbols $A \cong_{\mathfrak{n}} B$) if there exists bijective map map $\Phi : A \to B$ which respects the labelling, namely $\exists Q \in \mathcal{U},\ \forall \lambda \in Q$,
$$\Phi(A_\lambda) = B_\lambda$$

Comparison of the numerosities theory with the cardinal numbers

From the point (1), it follows straightforwardly the following theorem:

Theorem

If $\mathfrak{num}(A) < \mathfrak{num}(B)$, *then* $|A| \leq |B|$.

Comparison of the numerosities theory with the ordinal numbers

Now let us compare the set of numerosities with the ordinal numbers. In order to do this, we take an injective map

$$\Psi : \mathbf{Num} \to \mathbf{Ato}.$$

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Hence it makes sense to talk of the numerosity of a set of numerosities.

Definition of the ordinal numerosities

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Definition

The set **Ordn** of *ordinal numerosities* is defined by the following property:

$$\gamma \in \mathbf{Ordn} \Leftrightarrow \gamma = \mathfrak{num}\left(\Omega_\beta\right)$$

where

$$\Omega_\beta := \{x \in \mathbf{Ordn} \mid x < \beta\}$$

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2 &: = \mathfrak{num}\left(\{0,1\}\right) \in \mathbf{Ordn} \\
&\ldots\ldots\ldots\ldots \\
n+1 &: = \mathfrak{num}\left(\{0,..,n\}\right) \\
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\omega &: = \mathfrak{num}\left(\mathbb{N}\right), \quad \mathbb{N} := \{0,1,..,n,\ldots\} \\
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and so on. Thus, we get a "copy" of the Von Nemann ordinal numbers.

Comparison of the numerosities theory with the ordinal numbers

Clearly

$$\mathbf{Ordn} \subset \mathbf{Num}$$

since

$$\omega - 1 = \mathfrak{num}\left(\mathbb{N}^{+}\right) = \mathfrak{num}\left(\{1, 2.., n, ...\}\right) \notin \mathbf{Ordn}.$$

Cardinal numbers, ordinal numbers and numerosities

It is well known that the cardinal numbers can be identified with some ordinal numbers by the map

$$\Psi : \mathbf{Card} \rightarrow \mathbf{Ord}$$

defined by

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Similarly, we can identify the cardinal numbers with some numerosities by the map

$$\Psi : \mathbf{Card} \to \mathbf{Num}$$

defined by

$$\Psi_{\mathfrak{n}}(\beta) := \min\left\{x \in \mathbf{Ordn} \mid |\Omega_x| = \beta\right\}$$

Cardinal numbers, ordinal numbers and numerosities

In conclusion, we have the following diagram:

$$
\begin{array}{ccccccc}
\mathbb{N} & \subset & \mathbf{Card} & \subset & \mathbf{Ord} & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbb{N} & \subset & \mathbf{Cardn} & \subset & \mathbf{Ordn} & \subset & \mathbf{Num}
\end{array}
$$

where $\mathbf{Cardn} := \Psi_{\mathfrak{n}}(\mathbf{Card})$ is the set of cardinal numerosities and the vertical arrows are isomorphisms.

Basic properties of numerosities

For every set of labels $\mathfrak{L}$ the following result holds:

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$$\mathfrak{F}_\omega\left(X, E\right) := \left\{f : D \to E \mid D \in \wp_\omega(X)\right\}$$

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4. **Hessenberg principle**: *the operation "$+$" and "$\cdot$" between ordinal numerosities correspond to the natural (Hessenberg) operations between ordinals.*

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5. $\mathfrak{num}(\mathbb{Q}^+) = \alpha^2;$

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and $E \subset \mathbb{R} \subset$ **Ato** *is a Lebesgue measurable set, then*

$$m_L\left(E\right) = st\left(\frac{\mathfrak{num}\left(E\right)}{\beta}\right),$$

where m_L *denotes the Lebesgue measure.*

Open question

We will end this talk with an open question worth to think about:

We will end this talk with an open question worth to think about:

If we give up the Cartesian Product Principle *and we consider only the product* "$\otimes$", *there is a numerosity counting system* $(\Lambda, \mathbf{Num}, \mathfrak{num})$ *where* Λ *is the class of all sets?*

Thank you for your attention!

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Happy birthday, Mauro!