EUCLIDEAN INTEGERS, EUCLIDEAN ULTRAFILTERS, AND EUCLIDEAN NUMEROSITIES

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The Euclidean integers

We present an axiomatization of the ordered domain $\mathbf{Z}_\kappa$ of the Euclidean integers: from the algebraic point of view, the Euclidean integers are a *non-Archimedean discretely ordered superring $\mathbf{Z}_\kappa$ of $\mathbb{Z}$*, with a supplementary structure, the *Euclidean structure*, introduced axiomatically *via* the *transfinite sum*

$$\sum_{\alpha} a_\alpha = \mathbf{\Sigma}(\mathbf{a}), \quad \text{where} \quad \langle a_\alpha \mid \alpha < \kappa \rangle = \mathbf{a} \in \mathbb{Z}^\kappa,$$

Remark that we intend that any transfinite sum comprehends *all summands* a_α, $\alpha < \kappa$. When needed, we restrict the sum to a subset $K \subseteq \kappa$ by means of the characteristic function of K

$$\sum_{\alpha \in K} a_\alpha = \sum_{\alpha} b_\alpha, \ \text{ with } \ b_\alpha = a_\alpha \cdot \chi_K(\alpha), \text{ and } \ \chi_K(\alpha) = \begin{cases} 1 & \text{if } \alpha \in K, \\ 0 & \text{otherwise.} \end{cases}$$

The square inclusion $\sqsubset$ between ordinals

Given the *base 2 normal form* of the ordinal $\alpha = 2^{\alpha_1} \oplus \ldots \oplus 2^{\alpha_n}$, where $\oplus$ denotes the so called *natural commutative ordinal sum*, the *associated finite set of ordinals* is the set $\ell_\alpha = \{\alpha_1, \ldots, \alpha_n\}$, and, coherently, 0 corresponds to the empty set $\ell_0 = \emptyset$.

The *square inclusion* relations $\sqsubset$, $\sqsubseteq$ on ordinals are defined by

$\alpha \sqsubset \beta \Longleftrightarrow \ell_\alpha \subset \ell_\beta, \quad \alpha \sqsubseteq \beta \Longleftrightarrow \ell_\alpha \subseteq \ell_\beta \quad (hence\ \alpha \sqsubset \beta \Rightarrow \alpha < \beta)$;

the corresponding *supremum* is $\bigvee_1^n \alpha_i = \alpha \ \Leftrightarrow \ \ell_\alpha = \bigcup_1^n \ell_{\alpha_i}$

Proposition. *For an infinite set of ordinals A, let $\widehat{A} = \{\beta \mid \ell_\beta \subseteq A\}$: then A is $\sqsubset$-cofinal in $\widehat{A} \Longleftrightarrow A$ is $\sqsubset$-directed, and then the map $\ell : \alpha \mapsto \ell_\alpha$ is an isomorphism of $(\widehat{A}, \sqsubset)$ onto $([A]^{<\omega}, \subset) \cong (|A|, \sqsubset)$*

In particular $\kappa \geq \aleph_0 \implies \kappa = \hat{\kappa} \implies (\kappa, \sqsubset) \cong ([\kappa]^{<\omega}, \subset)$.

The Axioms of the Euclidean integers

This axiomatization is similar to that of the Euclidean numbers of [3], but introduces the new stronger axiom (SRA), which makes the semiring $\mathbf{Z}_{\kappa}^{\geq 0}$ of the nonnegative Euclidean integers coincide with the set of all transfinite sums of natural numbers. We make the natural assumption that a *transfinite sum* coincides with the *ordinary sum* of the ring $\mathbf{Z}_{\kappa}$ when the number of *non-zero summands* is *finite*.

In the following axioms, for sake of clarity, we denote general Euclidean numbers by *fractures* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{s}, \mathfrak{t}, \mathfrak{x}, \mathfrak{y}, \mathfrak{z}$, ordinary integers by *latin* letters $a, b, c, m, n, p, q, u, v, w, x, y, z$, κ-sequences by boldface letters $\mathbf{a}, \mathbf{b}, \mathbf{n}, \mathbf{x}, \mathbf{y}, \mathbf{z}$, and ordinals by greek letters $\alpha, \beta, \gamma, \delta, \eta$, with κ, ν, μ reserved for cardinals.

(RA) **Representation Axiom**: For every $\mathfrak{x}$ in $\mathbf{Z}_\kappa$ there exists a κ-sequence $\mathbf{x} = \langle x_\alpha \mid \alpha < \kappa \rangle \in \mathbb{Z}^\kappa$ such that $\mathfrak{x} = \sum_\alpha x_\alpha$.

(LA) **Linearity Axiom**: The transfinite sum is $\mathbb{Z}$-linear, *i.e.*

$$u\textstyle\sum_\alpha x_\alpha + v\sum_\alpha y_\alpha = \sum_\alpha (ux_\alpha + vy_\alpha) \quad \textit{for all} \quad u, v, x_\alpha, y_\alpha \in \mathbb{Z}.$$

(CA) **Comparison Axiom** For all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^\kappa$

$$\exists\, \theta < \kappa \ \forall\, \delta \sqsupseteq \theta \Big(\sum_{\alpha \sqsubseteq \delta} x_\alpha \leq \sum_{\alpha \sqsubseteq \delta} y_\alpha \Big) \Longrightarrow \sum_\alpha x_\alpha \leq \sum_\alpha y_\alpha$$

(Remark that $\sum_{\alpha \sqsubseteq \delta} x_\alpha$ is a *finite sum of ordinary integers*.)

(PA) **Product axiom**: For all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^\kappa$

$$\Big(\sum_\alpha x_\alpha\Big)\Big(\sum_\beta y_\beta\Big) = \sum_{\alpha,\beta} x_\alpha y_\beta = \sum_\gamma \Big(\sum_{\alpha \vee \beta = \gamma} x_{\alpha\beta} \Big),$$

Remark that $\sum_{\gamma \sqsubseteq \delta} \sum_{\gamma = \alpha \vee \beta} x_{\alpha\beta} = \sum_{\alpha,\beta \sqsubseteq \delta} x_{\alpha\beta}$ is a *finite sum of integers*, and the corresponding comparison criterion holds:

$$\exists\, \theta < \kappa \ \forall\, \delta \sqsupseteq \theta \Big(\sum_{\alpha,\beta \sqsubseteq \delta} x_{\alpha\beta} \leq \sum_{\alpha,\beta \sqsubseteq \delta} y_{\alpha\beta} \Big) \Longrightarrow \sum_{\alpha,\beta} x_{\alpha\beta} \leq \sum_{\alpha,\beta} y_{\alpha\beta}$$

The Strong Representation Axiom

(SRA) **Strong Representation Axiom**: *For all* $\mathfrak{x} \geq 0$ *in* $\mathbf{Z}_\kappa$ *there exists* $\mathbf{n} = \langle n_\alpha \mid \alpha < \kappa \rangle \in \mathbb{N}^\kappa$ *such that* $\mathfrak{x} = \sum_\alpha n_\alpha$.

The axiom (SRA) directly yieds the representation axiom (RA), since every Euclidean integer is the *difference* of two disjoint transfinite sums of natural numbers

$$\mathfrak{z} = \sum_{\alpha \in A^+} z_\alpha - \sum_{\alpha \in A^-} |z_\alpha|, \quad \text{where } A^\pm = \{\alpha \mid z_\alpha \gtrless 0\}.$$

More important, the strong axiom (SRA) has the interesting consequence that all Euclidean integers are of the simpler form

$$\mathfrak{z} = \pm \sum_\alpha n_\alpha \text{ for some } \mathbf{n} \in \mathbb{N}^\kappa.$$

This fact is of great importance for the theory of numerosities.

The axiom (CA) yields the useful property

Translation invariance: Let $\eta, \gamma < \kappa$ and $x_\alpha = 0$ for $\alpha \geq 2^\eta$:

$$\text{(TI)} \qquad \sum_\alpha x_\alpha = \sum_\delta y_\delta, \ \text{ where } \ y_\delta = \begin{cases} x_\alpha & if \ \ \delta = 2^\eta \gamma + \alpha \\ 0 & otherwise \end{cases}.$$

In fact, $\sum_{\alpha \sqsubseteq \beta} x_\alpha = \sum_{\delta = 2^\eta \gamma + \alpha \sqsubseteq \beta} y_\delta$, for $\alpha < 2^\eta$ and $\beta \sqsupseteq 2^\eta \gamma$.

A stronger interesting property is the following

Double sum linearization: Let $x_{\beta,\gamma} = 0$ for $\gamma, \beta \geq 2^\eta$, then

$$\text{(DSL)} \ \sum_{\beta,\gamma} x_{\beta\gamma} = \sum_\alpha y_\alpha, \ \text{ where } \ y_\alpha = \begin{cases} x_{\beta\gamma} & if \ \ \alpha = 2^\eta \beta + \gamma \\ 0 & otherwise \end{cases}.$$

In order to get (DSL), one has to strengthen (CA) by putting

$$D(\eta, \theta) = \{\delta = 2^{\eta \cdot 2}\alpha + 2^\eta \xi + \xi \mid \xi < 2^\eta, \, \alpha < \kappa, \delta \sqsupseteq \theta\}$$

and postulating, for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^\kappa$,

$$\exists\, \eta, \theta < \kappa \ \forall\, \delta \in D(\eta, \theta) \left(\sum_{\alpha \sqsubseteq \delta} x_\alpha \leq \sum_{\alpha \sqsubseteq \delta} y_\alpha \right) \Longrightarrow \sum_\alpha x_\alpha \leq \sum_\alpha y_\alpha$$

Embedding the ordinals

The properties (TI) and (DSL) have the interesting consequence that, by taking the transfinite sum of their characteristic functions, the ordinals less than κ, with the so called "natural sum and product" $\oplus, \otimes$, become an ordered subsemiring of $\mathbf{Z}_\kappa$.

Theorem. *Put* $\Psi(\beta) = \sum_\alpha \chi_\beta(\alpha)$, *where* $\chi_\beta(\alpha) = \begin{cases} 1 & \text{if } \alpha < \beta, \\ 0 & \text{otherwise} \end{cases}$.

Then $\alpha < \beta \iff \Psi(\alpha) < \Psi(\beta)$, *and* $\Psi(\alpha \oplus \beta) = \Psi(\alpha) + \Psi(\beta)$, *hence* Ψ *is an isomorphic embedding of* $(\kappa; <, \oplus)$, *as ordered semigroup, into the nonnegative part of* $\mathbf{Z}_\kappa$.

If moreover (DSL) *holds, then also* $\Psi(\alpha \otimes \beta) = \Psi(\alpha) \cdot \Psi(\beta)$, *and* Ψ *is an isomorphic embedding of* $(\kappa; <, \oplus, \otimes)$, *as ordered semiring, into the nonnegative part of the ring* $\mathbf{Z}_\kappa$.

The counting functions

The *counting function* of a κ-sequence of integers $\mathbf{x} \in \mathbb{Z}^\kappa$ is

$$\mathbf{f_x} : \kappa \to \mathbb{Z} \text{ defined by } \mathbf{f_x}(\alpha) = \textstyle\sum_{\beta \sqsubseteq \alpha} x_\beta.$$

Lemma. *Every function $\psi : \kappa \to \mathbb{Z}$ is the counting function $\mathbf{f_x}$ of some κ-sequence $\mathbf{x} \in \mathbb{Z}^\kappa$*, e.g. *put*

$$x_0 = \psi(0), \quad x_\alpha = \psi(\alpha) - \sum_{\beta \sqsubset \alpha} x_\beta, \quad \textit{thus} \quad \psi(\alpha) = \sum_{\beta \sqsubseteq \alpha} x_\beta.$$

Call *fine* a filter on κ containing all *cones* $C(\alpha) = \{\beta \mid \alpha \sqsubseteq \beta\}$, for $\alpha < \kappa$, and let $\mathcal{U}$ be any fine ultrafilter containing the zero-sets of all counting functions, *i.e.* the sets $\mathcal{Z}(\mathbf{x}) = \{\alpha < \kappa \mid \mathbf{f_x}(\alpha) = 0\}$, for $\mathbf{x} \in \mathbb{Z}^\kappa$.

Then we have the following ultrapower characterization ($\mathcal{U}$ must be fine, in order to get axiom (CA)).

The ultrapower characterization of $\mathbf{Z}_\kappa$

Theorem. *Let* $\mathcal{U}$ *be a fine ultrafilter over* κ*, containing all zero-sets* $\mathcal{Z}(\mathbf{x}) = \{\alpha < \kappa \mid \mathbf{f}_{\mathbf{x}}(\alpha) = 0\}$, $\mathbf{x} \in \mathbb{Z}^\kappa$*. Let* $\mathfrak{p}$ *be the prime ideal of the ring* $\mathbb{Z}^\kappa$ *corresponding to* $\mathcal{U}$*, and let* $\pi_{\mathcal{U}} : \mathbb{Z}^\kappa \to \mathbb{Z}^\kappa_{\mathcal{U}}$ *and* $\pi_{\mathfrak{p}} : \mathbb{Z}^\kappa \to \mathbb{Z}^\kappa/\mathfrak{p}$ *be the canonical projections. Then there exist unique isomorphisms* $\sigma : \mathbf{Z}_\kappa \to \mathbb{Z}^\kappa/\mathfrak{p}$, $\varphi : \mathbb{Z}^\kappa/\mathfrak{p} \to \mathbb{Z}^\kappa_{\mathcal{U}}$ *that make the following diagram commute:*

$$
\begin{array}{ccccc}
\mathbb{Z}^\kappa & & \xrightarrow{\quad \mathbf{f} \quad} & & \mathbb{Z}^\kappa \\
\Big\downarrow \Sigma & & \searrow \pi_{\mathfrak{p}} & & \Big\downarrow \pi_{\mathcal{U}} \\
\mathbf{Z}_\kappa & \xrightarrow{\ \sigma\ } & \mathbb{Z}^\kappa/\mathfrak{p} & \xrightarrow{\ \varphi\ } & \mathbb{Z}^\kappa_{\mathcal{U}}
\end{array}
$$

($\mathbf{f}$ *and* Σ *map each* $\mathbf{x} \in \mathbb{Z}^\kappa$ *to its counting function* $\mathbf{f}_{\mathbf{x}} \in \mathbb{Z}^\kappa$ *and respectively to its transfinite sum* $\sum(\mathbf{x}) \in \mathbf{Z}_\kappa$*)*

Validating the axiom (SRA)

Any *fine ultrafilter* $\mathcal{U}$ on κ validates the four axioms (RA),(LA), (CA), and (PA), but the ring $\mathbf{Z}_\kappa$ satisfies the strong representation axiom (SRA) if and only if Σ maps the semiring $\mathbb{N}^\kappa$ of all κ-sequences of natural numbers onto the nonnegative part $\mathbf{Z}_\kappa^{\geq 0}$ of $\mathbf{Z}_\kappa$. Equivalently, given a function $\psi \in \mathbb{Z}^\kappa$ that is positive modulo $\mathcal{U}$, the axiom (SRA) postulates the existence of a κ-sequence $\mathbf{x} \in \mathbb{N}^\kappa$ such that

$$\{\alpha < \kappa \mid \mathbf{f_x}(\alpha) = \textstyle\sum_{\beta \sqsubseteq \alpha} x_\beta = \psi(\alpha)\} \in \mathcal{U}.$$

Now, nonnegative κ-sequences $\mathbf{x}$ give rise to nondecreasing counting functions $\mathbf{f_x}$, and conversely, so one needs a fine ultrafilter $\mathcal{U}$ that includes, for each $\psi \in \mathbb{N}^\kappa$, a set U_ψ such that

(#) $$\forall \alpha, \beta \in U_\psi \left(\alpha \sqsubset \beta \implies \psi(\alpha) \leq \psi(\beta)\right).$$

The Euclidean ultrafilters

Let $[\kappa]^2_{\sqsubset} = \{(\alpha, \beta) \mid \alpha \sqsubset \beta\}$ be the set of all $\sqsubset$-ordered pairs: for $\psi \in \mathbb{N}^\kappa$, define the partition

$$G_\psi : [\kappa]^2_{\sqsubset} \to \{0, 1\} \text{ by } G_\psi(\alpha, \beta) = \begin{cases} 0 & \text{if } \psi(\alpha) > \psi(\beta), \\ 1 & \text{otherwise.} \end{cases}$$

Clearly, the partition G_ψ does not admit any $\sqsubset$-increasing 0-homogeneous sequence (call such a sequence a 0-*chain*).

Call *Euclidean* a fine ultrafilter $\mathcal{U}$ on κ if for all $\psi \in \mathbb{N}^\kappa$ the partition G_ψ of $[\kappa]^2_{\sqsubset}$ has a homogeneous set $U_\psi \in \mathcal{U}$, which, being $\sqsubset$-cofinal, cannot be 0-homogeneous, so it satisfies (#). Then

Corollary. *The ring* $\mathbf{Z}_\kappa \cong \mathbb{Z}^\kappa_{\mathcal{U}}$ *satisfies the strong representation axiom* (SRA) *if and only if the ultrafilter* $\mathcal{U}$ *is Euclidean.* □

We are left with the question of the existence of Euclidean ultrafilters.

The partition property $[A]^{<\omega}_{\subset} \to (cofin)^r_k$

The partition property $[A]^{<\omega}_{\subset} \to (cofin)^r_k$ says that any finite partition of all $\subset$-ordered r-tuples of finite subsets of A admits a $\subset$-cofinal homogeneous subset $H \subseteq A$, *i.e.* every $u \in [A]^{<\omega}$ is included in some $v \in [H]^{<\omega}$, and all $\subset$-ordered r-tuples from $[H]^{<\omega}$ belong to the same piece of the partition (see [9]).

Clearly, the partition property $[A]^{<\omega}_{\subset} \to (cofin)^2_k$ depends only on $|A|$. Considering the corresponding notion for the square inclusion $\sqsubset$ on κ, the property $\kappa_{\sqsubset} \to (\sqsubset\text{-}cofin)^2_2$ would yield directly the existence of Euclidean ultrafilters on κ, but its validity for $\kappa > \aleph_1$ is still open, half a century after [9]. (For $\kappa = \aleph_0$ it follows immediately from Ramsey's Theorem, while for $\kappa = \aleph_1$ it is proved in [10].)

The partition property $\kappa \to (\omega, cofin)^2_{\sqsubset}$

However a (weaker) property, stating an appropriate version of the Erdös-Dushnik-Miller partition property $\kappa \to (\omega, \kappa)^2$ suffices.

The partition property $\kappa \to (\omega, cofin)^2_{\sqsubset}$ affirms that *any 2-partition $G : [\kappa]^2 \to \{0, 1\}$ that does not admit a 0-chain* (*i.e.* a 0-homogeneous $\sqsubset$-increasing sequence), has a $\sqsubset$-cofinal homogeneous set H (necessarily 1-homogeneous).

This property suffices to get Euclidean ultrafilters, namely

Lemma 1. *If $\kappa \to (\omega, \kappa)^2_{\sqsubset}$ holds, then there are Euclidean ultrafilters on κ.* □

In [7] the validity of $\kappa \to (\omega, cofin)^2_{\sqsubset}$ is proved for all uncountable cardinals, hence the existence of rings of Euclidean integers $\mathbf{Z}_\kappa$ satisfying the strong representation axiom (SRA) is granted.

The Euclidean common notions

A satisfactory notion of *measure of size* should abide by the famous five *common notions* of Euclid's Elements, which traditionally embody the properties of any kind of *magnitudines*:

(E1) *Things equal to the same thing are also equal to one another.*

(E2) *And if equals be added to equals, the wholes are equal.*

(E3) *And if equals be subtracted from equals, the remainders are equal.*

(E4) *Things* [exactly] applying onto *one another are equal to one another.**

(E5) *The whole is greater than the part.*

* Here we translate $\epsilon\phi\alpha\rho\mu o\zeta o\nu\tau\alpha$ by "[exactly] applying onto", instead of the usual "coinciding with". As pointed out by T.L. Heath in his commentary [**?**], this translation seems to give a more appropriate rendering of the mathematical usage of the verb $\epsilon\phi\alpha\rho\mu o\zeta\epsilon\iota\nu$.

Measureing the size of sets

The usual measure of the size of sets is the classical Cantorian *"cardinality"*, grounded on the so called *Hume's Principle "Two sets have the same size if and only if there exists a biunique correspondence between them."*

This assumption might seem natural, and even *implicit in the notion of counting*; but it strongly violates the equally natural *Euclid's principle A set is greater than its proper subsets,* which in turn seems *implicit in the notion of magnitudo*.

- Call a measure of size $\mathfrak{k}$ for sets *Cantorian* if, for all A, B:

  (HP) $\qquad \mathfrak{k}(A) = \mathfrak{k}(B) \iff \exists f : A \to B$ *biunique*

- Call a measure of size $\mathfrak{n}$ for sets *Euclidean* if, for all A, B:

  (EP) $\mathfrak{n}(A) < \mathfrak{n}(B) \iff \exists A' \subset B' \big(\mathfrak{n}(A) = \mathfrak{n}(A'), \mathfrak{n}(B') = \mathfrak{n}(B)\big)$

  (Remark the use of *proper inclusion*)

Euclidean size of sets (numerosity)

The consistency of the principle (EP) for *uncountable sets* appeared problematic from the beginning, and this question has been posed in several papers (see [1, 2, 5]), where only the *literal set-theoretic translation* of the fifth Euclidean notion has been obtained, *i.e.* the sole left pointing arrow of (EP):

(E5) $$A \subset B \implies \mathfrak{n}(A) < \mathfrak{n}(B)$$

(On the other hand, it is worth recalling that also the totality of the Cantorian weak cardinal ordering had to wait more than two decades till *Zermelo's new axiom of choice* to be established!)

A general discussion of different ways for comparing and measuring the size of sets can be found in [8].

Addition of numerosities

One wants not only *compare*, but also *add and subtract numerosities*, and the following *Aristotle's Principle*†

(AP) $$\mathfrak{n}(A) = \mathfrak{n}(B) \iff \mathfrak{n}(A \setminus B) = \mathfrak{n}(B \setminus A)$$

is convenient, because it yields both the second and third Euclidean common notions, with the natural definitions of *addition* as *(disjoint) union*, and *subtraction* as *(relative) complement*

$$\mathfrak{n}(A) + \mathfrak{n}B = \mathfrak{n}(A \cup B) \quad for\ all \quad A, B\ such\ that \quad A \cap B = \emptyset$$

Moreover, asuming (AP), the full principle (EP) follows by the particular case of the empty set

(E0) $$\mathfrak{n}(A) = \mathfrak{n}(\emptyset) \iff A = \emptyset$$

† This priciple has been named *Aristotle's Principle* in [4, **?**], because it resembles Aristotle's preferred example of a "general axiom". It is especially relevant in this context, because (AP) implies both the second and the third Euclidean common notions, and also the fifth whenever no nonempty set is equivalent to $\emptyset$, as stated in the proposition below.

The Subtraction Principle

In particular one gets from (AP)+(EP) the *"most wanted Subtraction Principle"* of [1]:

(diff) $\mathfrak{n}(A) < \mathfrak{n}(B) \iff \exists C \left(C \cap A = \emptyset \text{ and } \mathfrak{n}(C \cup A) = \mathfrak{n}(B) \right)$

The consistency problem of the Subtraction Principle, studied in several papers dealing with Euclidean (also called *Aristotelian*) notions of size for sets, receeved a positive answer only for *countable sets* in [5, 4], thanks to the use of *selective ultrafilters*. A positive answer for sets of *arbitrary cardinality* is obtained in [8], where it follows from the existence of *Euclidean ultrafilters*, equivalent to the consistency of the axiom (SRA).

Multiplication of numerosities

In classical mathematics, a *multiplicative* version of Euclid's second common notion *"if equals be multiplied by equals, the products are equal"* was never considered for "dishomogeneous" magnitudes, like geometric figures having different dimensions. However it seems natural to consider *abstract sets* as *homogeneous mathematical objects,* and a satisfying *arithmetic of numerosities* needs a *product*, with a corresponding *unit*. The Cantorian choice of *Cartesian products* and *singletons* makes any $A \times \{b\}, b \in B$ a *disjoint equinumerous copy of* A, thus making their union $A \times B$ the sum of "B-many copies of A", in accord with the intuitive idea of product.‡

‡*CAVEAT*: Cartesian products are optimal when any A, B are *multipliable* in the sense that $A \times B \cap A \cup B = \emptyset$, but not when singletons cannot be unitary for a Euclidean theory, *e.g.* when *transitive universes* are considered : $V_\omega \times \{x\} \subset V_\omega$ for any $x \in V_\omega$. (see [1, 8]).

Natural congruences

Once the general Hume's principle cannot be assumed, the fourth Euclid's common notion

Things [exactly] applying onto one another are equal to one another

is left in need of an adequate choice of natural *congruences,* i.e. *size-preserving "exact applications"*. When dealing with sets of tuples, a natural choice seems to take those applications τ that preserve the *support* (the set of components) of the tuple:

$$\tau(a_1, \dots, a_k) = \tau(b_1, \dots, b_n) \iff \{a_1, \dots, a_k\} = \{b_1, \dots, b_n\}$$ and

postulate the following *Congruence Principle*

(CP) $\mathfrak{n}(\tau[A]) = \mathfrak{n}(A)$ for all support preserving τ,[§]

[§] So in particular, although the Cartesian product is neither commutative nor associative *stricto sensu*, the product of numerosities is.

Euclidean numerosities as Euclidean integers

The properties of the ring $\mathbf{Z}_\kappa$ of the Euclidean integers allow for assigning a Euclidean numerosity to "Punktmengen", *i.e.* sets of tuples, over any line $\mathbb{L}$ of arbitrary size κ.

Definition. A *(Euclidean) numerosity* for "Punktmengen" over $\mathbb{L}$ is a function $\mathfrak{n} : \mathbb{W} \to \mathbf{Z}_\kappa$, with a set $\mathbb{W} \subseteq \mathcal{P}(\bigcup_{n\in\mathbb{N}} \mathbb{L}^n)$ such that

$$A \cup B,\ A \times B \in \mathbb{W} \iff A, B \in \mathbb{W}, \qquad C \subseteq A \in \mathbb{W} \Longrightarrow C \in \mathbb{W},$$

and the following conditions are satisfied for all $A, B, C \in \mathbb{W}$:

(EP) $\mathfrak{n}(A) < \mathfrak{n}(B) \iff \exists B'(A \subset B',\ \mathfrak{n}(B') = \mathfrak{n}(B))$,

(AP) $\mathfrak{n}(A) = \mathfrak{n}(B) \iff \mathfrak{n}(A \setminus B) = \mathfrak{n}(B \setminus A)$;

(PP) $\mathfrak{n}(A) = \mathfrak{n}(B) \iff \mathfrak{n}(A \times C) = \mathfrak{n}(B \times C)$ (for all $C \neq \emptyset$);

(UP) $\mathfrak{n}(A) = \mathfrak{n}(A \times \{w\})$ for all $w \in W = \bigcup \mathbb{W}$.

(CP) $\mathfrak{n}(\tau[A]) = \mathfrak{n}(A)$ for all *support-preserving* bijections τ;

Since in a general set-theoretic context there are no "geometric" or "analytic" properties to be considered, the sole relevant characteristic of the line $\mathbb{L}$ remains *cardinality*, so a convenient choice seems to be simply identify $\mathbb{L}$ *with its cardinal* κ, thus obtaining the fringe benefit that *no pair of ordinals is an ordinal, and Cartesian products may be freely used*.

Moreover this numerosity might be extended to the whole universe by suitably labelling each set by ordinals, under simple set theoretic assumptions, *e.g.* Von Neumann's axiom, that gives a (class-)bijection between the universe V and the class Ord of all ordinals.

The generalized characteristic functions

Start with $\mathfrak{n}:\mathcal{P}(\kappa)\to\mathbf{Z}_\kappa$ as the transfinite sum of the characteristic functions of each subset of κ:

$$\mathfrak{n}(A)=\sum_\alpha \chi_A(\alpha),\quad where\quad \chi_A(\alpha)=\begin{cases}1 & \text{if } \alpha\in A\\ 0 & \text{otherwise.}\end{cases}$$

Then assign to each n-tuple $\overline{\alpha}=(\alpha_1,\ldots,\alpha_n)\in\kappa^n$ the ordinal $\psi_n(\overline{\alpha})=\alpha_1\vee\ldots\vee\alpha_n<\kappa$, and extend $\mathfrak{n}$ to $\mathcal{P}(\kappa^n)$ by

$$\mathfrak{n}(A)=\textstyle\sum_\alpha \chi_A^{(n)}(\alpha),\quad \text{where}\quad \chi_A^{(n)}(\alpha)=|\{\overline{\alpha}\in A\mid \psi_n(\overline{\alpha})=\alpha\}|$$

for all[¶] $A\subseteq\kappa^n$.

Finally extend $\mathfrak{n}$ to all *finite dimensional point sets*, *i.e.* sets A such that $\{n\mid A\cap\kappa^n\neq\emptyset\}$ is finite, by putting

$$\mathfrak{n}(A)=\sum_n \mathfrak{n}(A\cap\kappa^n).$$

¶ Remark that we are assigning the same ordinal α to $\alpha\in\kappa$, to $(\alpha_1,\alpha_2)\in\kappa^2$ if $\alpha=\alpha_1\vee\alpha_2$, ..., to $(\alpha_1,\ldots,\alpha_n)\in\kappa^n$ if $\alpha=\bigvee_1^n\alpha_i$, hence the functions $\chi^{(n)}$, for $n>1$, are not properly characteristic functions, but they assume nonnegative integer values, so their sums are nonnegative Euclidean integers.

Then the five Euclid's Common Notions are satisfied, all finite sets receive their number of elements as numerosity, and

- the principle (EP) is equivalent to the axiom (SRA); (both postulate that the difference $\chi_A - \chi_B$ of two (generalized) characteristic functions, when positive, has the same transfinite sum of a single positive function χ_C)
- the Aristotelian principle (AP) holds because
  $$\mathfrak{n}(A) = \mathfrak{n}(A \setminus B) + \mathfrak{n}(A \cap B),\ \mathfrak{n}(B) = \mathfrak{n}(B \setminus A) + \mathfrak{n}(A \cap B);$$
- the multiplicative principles (UP) and (PP) follow directly from the product axiom (PA);
- the congruence principles (CP) holds by the definition of the generalized characteristic funtions χ_A^n.

moreover every point set is equinumerous to a set of ordinals, and conversely every nonnegative Euclidean integer $\mathfrak{x}$ is the numerosity of a set X of ordinals.

The power of numerosities

The power $\mathfrak{m}^{\mathfrak{n}}$ of infinite numerosities is here always well-defined, since numerosities are *positive euclidean numbers*, hence *nonstandard natural numbers.* By using finite approximations given by intersections with *suitable finite sets,* the interesting relation

$$2^{\mathfrak{n}(X)} = \mathfrak{n}([X]^{<\omega}),$$

has been obtained already in [1]. Since the comparison axiom (CA) evaluates transfinite sums by finite sums $\sum_{\alpha \sqsubseteq \delta} \chi_A(\alpha)$, one obtains the following general set theoretic interpretation of powers:

$$\mathfrak{m}(Y)^{\mathfrak{n}(X)} = \mathfrak{n}(\{f : X \to Y \mid |f| < \aleph_0\}),$$

by considering sets of ordinals X, Y, and labelling each finite function f by the $\sqsubseteq$-supremum of the (finitely many) ordinals involved in f.

The interesting problem of finding appropriately defined arithmetic operations that give instead the numerosity of the *full powersets* and *function spaces* requires a quite different approach, and the history of the same problem for cardinalities suggests that it could not be properly solved.

The Weak Hume Principle

Perhaps the best way to view a Euclidean numerosity is looking at it as a *refinement* of Cantorian cardinality, able to separate sets that, although *equipotent*, should have in fact *really different sizes*, in particular when they are *proper subsets or supersets* of one another.
To this aim, the principle (EP) might be integrated by the "Weak Hume's Principle" postulating that *equinumerous sets are in a biunique correspondence*. Or better

(WHP) $$\mathfrak{n}(A) \leq \mathfrak{n}(B) \implies \exists f \text{ 1-to-1}, f : A \to B.$$

So the ordering of the Euclidean numerosities refines the cardinal ordering, and sums of ones of greater cardinality produce greater Euclidean integers.

This topic is dealt with in [8], where it is proved that the family of sets

$$Q^{>}_{AB} = \{\beta < \kappa \mid \textstyle\sum_{\alpha \sqsubseteq \beta} \chi_A(\alpha) > \sum_{\alpha \sqsubseteq \beta} \chi_B(\alpha)\} \text{ for} |A| > |B|$$

has the FIP together with the cones $C(\theta)$, hence may be contained in the fine ultrafilter $\mathcal{U}$. However it is not known whether that family might be included in an Euclidean ultrafilter, so the consistency of the weak Hume principle (WHP) with the difference property (diff) is still open.

The Subset Property

An interesting consequence of (WHP) combined with (EP) is the fact that *any initial segment* of $\mathbf{Z}^{\leq 0}$ generated by a numerosity $\mathfrak{n}(A)$ is in correspondence with $\mathcal{P}(A)$, so has size $2^{|A|}$, whereas in the large ultrapower models of [1, 2], one may have strictly increasing chains of sets of arbitrary length.