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Euclidean integers, Eucledean ultrafilters, and Euclidean numerosities

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We introduce axiomatically, for any cardinal κ, the ordered domain $\mathbf{Z}_\kappa$ of the *Euclidean integers*, characterized as the collection of the transfinite sums of all κ-sequences of integers. Most relevant is the *algebraic* characterization of the ordering: a Euclidean integer is *positive* if and only if it is *the transfinite sum of natural numbers.* The ring $\mathbf{Z}_\kappa$ is obtained by taking the *ultrapower of the ring of all finite partial sums* modulo special ultrafilters (also named *Euclidean*), here introduced to this end. (So $\mathbf{Z}_\kappa$ is actually a ring of *hyperintegers.*)

The existence of Euclidean ultrafilters follows from a new form of the Erdős-Dushmill-Miller partition property $\kappa \to (\omega, \kappa)$, namely $[A^{<\omega}]^2_\subset \to (\omega, \text{cofinal})^2_\subset$, meaning that *"for every 2-partition $G : [A^{<\omega}]^2_\subset \to \{0, 1\}$ of all $\subset$-comparable pairs of finite subsets of A, there exists a 0-homogeneous countable $\subset$-increasing sequence $C \subset A^{<\omega}$,or there exists a $\subset$-cofinal set $H \subset A^{<\omega}$ such that $[H]^2_\subset = \{(b, c) \in H^2 \mid b \subset c\}$ is 1-homogeneous for G."*

We call $\mathbf{Z}_\kappa$ the ring of the Euclidean integers because it arises in a theory of size of sets (*"numerosity"*), whose main aim is to save all the Euclidean common notions, including the fifth *"the whole is greater than the part"*, but still maintaining the Cantorian definitions of *ordering, addition and multiplication* of sets. Having at disposal transfinite sums of integers, the numerosity of a set may be the transfinite sums of its characteristic function, which is an Euclidean integer after an appropriate labelling of sets by ordinals. Then, in contrast to the awkward Cantorian cardinal arithmetic, numerosities are a positive semiring of *hypernatural numbers*, the non-negative part of the ring $\mathbf{Z}_\kappa$.

In particular, the algebraic order of $\mathbf{Z}_\kappa$ yields the *"difference property"* asserting that the difference of the numerosities of any two sets is itself the numerosity of some set. This property has been the "most wanted" property of the Euclidean theory of size since its origin at the beginning of the century. Moreover one obtains the supplementary benefits that every set is equinumerous to a set of ordinals, and conversely that every nonnegative Euclidean integer is the numerosity of a set X of ordinals, namely the transfinite sum of the characteristic function χ_X.

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