Prove Multidimensional van der Waerden's Theorem With A Simple Induction

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Logical Methods in Ramsey Theory and Related Topics

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OUTLINE:

1. Iterated elementary extensions
2. Proof of Ramsey's theorem
3. Proof of multidimensional van der Waerden's theorem

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Definition

Let $\mathfrak{A}_0$ be a structure with the base set A_0 containing all standard real numbers and all subsets of standard real numbers including $\mathbb{N}_0$ and $\mathbb{R}_0$. For example, let $\mathfrak{A}$ be the superstructure $(V(X); \in)$ truncated at level 100.

(a) *The structure $\mathfrak{A}'$ with base set A' is called an elementary extension of $\mathfrak{A}_0$, denoted by $\mathfrak{A}_0 \prec \mathfrak{A}'$, if there is a proper injection $i : A_0 \to A'$, called elementary embedding, such that*

$$\varphi(\bar{a}) \text{ is true in } \mathfrak{A}_0 \iff \varphi(\overline{i(a)}) \text{ is true in } \mathfrak{A}' \qquad (1)$$

for any first–order formula $\varphi(\bar{x})$ in the language of $\mathfrak{A}_0$ and any tuple $\bar{a}$ in A_0^n.

(b) *Let $\mathfrak{A}'$ and $\mathfrak{A}''$ be two elementary extensions of $\mathfrak{A}_0$. The relation $\mathfrak{A}' \prec \mathfrak{A}''$ and map $i : \mathfrak{A}' \prec \mathfrak{A}''$ can be defined similarly as in (a) with $\mathfrak{A}_0$ and $\mathfrak{A}'$ being replaced by $\mathfrak{A}'$ and $\mathfrak{A}''$.*

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We often write $\mathfrak{A}$ for a structure as well as the base set of the structure for notational convenience.

Proposition

There is a chain of elementary extensions

$$\mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \mathfrak{A}_2 \prec \cdots$$

such that the set $\mathbb{N}_m$ of all natural numbers in $\mathfrak{A}_m$ is an initial segment of the set $\mathbb{N}_{m+1}$ of all natural numbers in $\mathfrak{A}_{m+1}$ and there exist elementary embeddings $i_{m,n}$ from $\mathfrak{A}_n$ to $\mathfrak{A}_{n+1}$ for any $0 \leq m \leq n$ such that

(a) *$i_{m,n}(x) = x$ for $x \in \mathbb{N}_m$ and $i_{m,n} \restriction \mathfrak{A}_k = i_{m,k}$ for $m \leq k \leq n$,*

(b) *$i_{m,n}[\mathbb{N}_k \setminus \mathbb{N}_{k-1}] \subseteq \mathbb{N}_{k+1} \setminus \mathbb{N}_k$ for $k = m+1, \ldots, n$,*

(c) *$i_{m,n} \restriction \mathfrak{A}_k$ is an elementary embedding from $(\mathfrak{A}_k; \mathbb{R}_{k-l})$ to $(\mathfrak{A}_{n+1}; \mathbb{R}_{n+1-l})$ for $m \leq k \leq n$ and $0 < l \leq k - m$, where $(\mathfrak{A}_k; \mathbb{R}_{k-l})$ is the structure $\mathfrak{A}_k$ augmented by $\mathbb{R}_{k-l}$ as a unary relation.*

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Remark

(1) *Fix a non-principle ultrafilter $\mathcal{F}$ on $\mathbb{N}_0$. The elementary chain $\mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \mathfrak{A}_2 \prec \cdots$ can be constructed by iterating the ultrapower construction of $\mathfrak{A}_0$ modulo $\mathcal{F}$.*

(2) *There are different ways of doing iterations. Let $\mathcal{F}' = \mathcal{F}$. For example, to obtain $\mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \mathfrak{A}_2$, one can consider*

$$\left(\mathfrak{A}_0^{\mathbb{N}_0}/\mathcal{F}\right)^{\mathbb{N}_0}/\mathcal{F}' = \mathfrak{A}_1^{\mathbb{N}_0}/\mathcal{F}' = \mathfrak{A}_2,$$

which is called the external ultrapower construction, or

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Remark

(3) *By the external ultrapower construction one can produce an elementary embedding* $i_{0,1} : \mathfrak{A}_1 \to \mathfrak{A}_2$. *Note that* $i_{0,1}[\mathbb{N}_1 \setminus \mathbb{N}_0] \subseteq \mathbb{N}_2 \setminus \mathbb{N}_1$. *By the internal ultrapower construction one can produce another elementary embedding* $i_{1,1} : \mathfrak{A}_1 \to \mathfrak{A}_2$ *such that* $i_{1,1} \restriction \mathbb{N}_1$ *is an identity map.*

(4) *Since* $(\mathfrak{A}_1; \mathbb{R}_0)^{\mathbb{N}_0}/\mathcal{F}' = (\mathfrak{A}_2; \mathbb{R}_1)$, *the map* $i_{0,1}$ *is also an elementary embedding from* $(\mathfrak{A}_1; \mathbb{R}_0)$, *i.e., the model* $\mathfrak{A}_1$ *augmented with a new unary relation* $\mathbb{R}_0$, *to* $(\mathfrak{A}_2; \mathbb{R}_1)$, *i.e., the model* $\mathfrak{A}_2$ *augmented with a new unary relation* $\mathbb{R}_1$.

(5) *If one iterates internal ultrapowers of* $\mathfrak{A}_0$ m *times followed by external but* $\mathfrak{A}_m$*–internal ultrapowers* $n - m$ *times, one can obtain an elementary embedding* $i_{m,n} : \mathfrak{A}_n \to \mathfrak{A}_{n+1}$ *as stated in the proposition.*

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Theorem (Ramsey)

Given a coloring $c : [\mathbb{N}_0]^n \to [r]$ for some $r \in \mathbb{N}_0$, there exists an infinite set $A \subseteq \mathbb{N}_0$ such that $c \restriction [A]^n$ is a constant function.

Proof: Let $x_1 = [Id_{\mathbb{N}_0}]_{\mathcal{F}_0} \in \mathbb{N}_1 \setminus \mathbb{N}_0$ and let $x_{j+1} = i_{0,n}(x_j)$ for $j = 1, 2, \ldots, n-1$. Note that x_{j+1} is an equivalence class represented by $Id_{\mathbb{N}_j} : \mathbb{N}_j \to \mathbb{N}_j$. Let ${}^*c(\overline{x}) = c_0 \in [r]$ where $\overline{x} = \{x_1, x_2, \ldots, x_n\}$. We find an infinite set $A = \{a_1 < a_2 < \cdots\}$ in $\mathbb{N}_0$ by induction such that ${}^*c \restriction [A \cup \overline{x}]^n \equiv c_0$.

The basic case for $A = \emptyset$ is trivially true.

Assume that $A_m = \{a_1 < a_2 < \cdots < a_m\}$ has been constructed such that ${}^*c \restriction [A_m \cup \overline{x}]^n \equiv c_0$.

It suffices to find $a_{m+1} > a_m$ in $\mathbb{N}_0$ and $A_{m+1} = A_m \cup \{a_{m+1}\}$ such that ${}^*c \restriction [A_{m+1} \cup \overline{x}]^n \equiv c_0$.

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Proof: Let $x_1 = [Id_{\mathbb{N}_0}]_{\mathcal{F}_0} \in \mathbb{N}_1 \setminus \mathbb{N}_0$ and let $x_{j+1} = i_{0,n}(x_j)$ for $j = 1, 2, \ldots, n-1$. Note that x_{j+1} is an equivalence class represented by $Id_{\mathbb{N}_j} : \mathbb{N}_j \to \mathbb{N}_j$. Let ${}^*c(\overline{x}) = c_0 \in [r]$ where $\overline{x} = \{x_1, x_2, \ldots, x_n\}$. We find an infinite set $A = \{a_1 < a_2 < \cdots\}$ in $\mathbb{N}_0$ by induction such that ${}^*c \restriction [A \cup \overline{x}]^n \equiv c_0$.

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$$\varphi(i_{0,n}(A_m), i_{0,n}(x_1), i_{0,n}(x_2), \ldots, i_{0,n}(x_{n-1})):$$

$\exists x \in \mathbb{N}_1$ *greater than* a_m *such that*
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By the transfer principle, the sentence $\varphi(A_m, x_1, x_2, \ldots, x_{n-1})$ is true in $(\mathfrak{A}_{n-1}; \mathbb{R}_0)$, i.e., there is an $a_{m+1} > a_m$ in $\mathbb{N}_0$ such that ${}^*c \restriction [A_{m+1} \cup \{x_1, x_2, \ldots, x_{n-1}\}]^n \equiv c_0$ where $A_{m+1} = A_m \cup \{a_{m+1}\}$.

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Consider the elementary embedding $i_{p,n-1} : \mathfrak{A}_{n-1} \to \mathfrak{A}_n$. For each $b_j \in \overline{b}$, if $b_j < x_p$, then $i_{p,n-1}(b_j) = b_j$, and if $b_j = x_k > x_p$, then $i_{p,n-1}(x_{k-1}) = i_{0,n-1}(x_{k-1}) = x_k$. Let $\overline{b}' = i^{-1}_{p,n-1}(\overline{b})$. Then, $\overline{b}' \in [A_{m+1} \cup \{x_1, x_2, \ldots, x_{n-1}\}]^n$. Hence, ${}^*c(\overline{b}') = c_0$. So, ${}^*c(\overline{b}) = c_0$ by the elementarity of $i_{p,n-1}$. □

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The multidimensional van der Waerden's theorem is also called Gallai's theorem. Let $[n] := \{0, 1, \ldots, n-1\}$.

Fix a dimension s. A homothetic copy of $[n]^s$ is a set of the form

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for some $\vec{a} \in \mathbb{N}^s$ and positive $d \in \mathbb{N}$. We omit n in $HC_{\vec{a},d,n}$ after n is fixed.

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Proof: Fix $n \in \mathbb{N}_0$. Let $\lhd$ be the lexicographical order of $HC_{\vec{a},d}$. For each $0 \leq l < n^s$ let $HC_{\vec{a},d}(l)$ denote the l-th element of $HC_{\vec{a},d}$ under $\lhd$. Note that $HC_{\vec{a},d}(0) = \vec{a}$.

Let $\varphi_m(r, N)$ be the following first–order sentence:

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Note that the claim when $m = n^s - 1$ is the multidimensional van der Waerden's theorem.

Proof: Fix $n \in \mathbb{N}_0$. Let $\lhd$ be the lexicographical order of $HC_{\vec{a},d}$. For each $0 \leq l < n^s$ let $HC_{\vec{a},d}(l)$ denote the l-th element of $HC_{\vec{a},d}$ under $\lhd$. Note that $HC_{\vec{a},d}(0) = \vec{a}$.

Let $\varphi_m(r, N)$ be the following first–order sentence:

$$\forall c : [N]^s \to [r] \ \exists HC_{\vec{a},d} \subseteq [N]^s \ \exists c_0 \in [r] \left(c(HC_{\vec{a},d}(l)) = c_0 \ \text{ for } \ l = 0, 1, \ldots, m \right). \tag{2}$$

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Note also that $N \in \mathbb{N}_0$ depends on r, and if $N' > N$, then $\varphi_m(r, N)$ implies $\varphi(r, N')$. Hence, if N is hyperfinite, then $(\forall r \in \mathbb{N}_0)\, \varphi_m(r, N)$ is true.

It suffices to prove the claim by induction on $m \leq n^s - 1$. Call $HC_{\vec{a},d}$ in (2) monochromatic up to m with respect to c.

Proof of Claim (1): The case for $m = 0$ is trivial.

Assume that the claim is true for $m - 1$. We prove that the claim is true for $m < n^s$.

Given $r \in \mathbb{N}_0$, the task now is to find $N \in \mathbb{N}_0$ such that $\varphi_m(r, N)$ is true in $\mathfrak{A}_0$.

Work within $\mathfrak{A}_{r+1}$. Choose any $N_r \in \mathbb{N}_{r+1} \setminus \mathbb{N}_r$. It suffices to prove that $\varphi_m(r, 2N_r)$ is true in $\mathfrak{A}_r$ by the transfer principle.

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For any $\vec{x}, \vec{y} \in [N_r]^s$ we say that $\vec{x}$ and $\vec{y}$ have the same $2N_j$-type if $c(\vec{x} + \vec{z}) = c(\vec{y} + \vec{z})$ for any $\vec{z} \in [2N_j]^s$.

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Since the first–order sentence

$$(\forall r' \in \mathbb{N}_0)\,(\forall N \in \mathbb{N}_1 \setminus \mathbb{N}_0)\,\varphi_{m-1}(r', N)$$

is true in $(\mathfrak{A}_1; \mathbb{R}_0)$, the sentence

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In particular, $\varphi_{m-1}(r^{(2N_{j-1})^s}, N_j)$ is true in $\mathfrak{A}_{j+1}$ for $j = 1, 2, \ldots, r$.

Since the number of $2N_{j-1}$-types is at most $r^{(2N_{j-1})^s}$, we can find $HC_{\vec{a}_j, d_j} \subseteq [N_j]^s$ such that $HC_{\vec{a}_j, d_j}$ is monochromatic up to $m-1$ with respect to $2N_{j-1}$-types, i.e., $HC_{\vec{a}_j, d_j}(l)$ for $l = 0, 1, \ldots, m-1$ have the same $2N_{j-1}$-type.

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Define $HC_{\vec{a},d} \oplus HC_{\vec{a}',d'} := HC_{\vec{a}+\vec{a}',d+d'}$. Clearly, for any $l < n^s$ we have

$$(HC_{\vec{a},d} \oplus HC_{\vec{a}',d'})(l) = HC_{\vec{a},d}(l) + HC_{\vec{a}',d'}(l).$$

For each $j = 0, 1, \ldots, r$ let

$\vec{y}_j := HC_{\vec{a}_r,d_r}(0) + \cdots + HC_{\vec{a}_j,d_j}(0) + HC_{\vec{a}_{j-1},d_{j-1}}(m) + \cdots + HC_{\vec{a}_0,d_0}(m)$.

There must exist $0 \le j_1 < j_2 \le r$ such that $c(\vec{y}_{j_1}) = c(\vec{y}_{j_2})$. Let

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$$\begin{aligned} D := HC_{\vec{a}_r,d_r}(0) + \cdots + HC_{\vec{a}_{j_2},d_{j_2}}(0) \\ +HC_{\vec{a}_{j_2-1}} \oplus \cdots \oplus HC_{\vec{a}_{j_1},d_{j_1}} \\ +HC_{\vec{a}_{j_1-1},d_{j_1-1}}(m) + \cdots + HC_{\vec{a}_0,d_0}(m). \end{aligned}$$

Then D is a homothetic copy of $[n]^s$.

Define $HC_{\vec{a},d} \oplus HC_{\vec{a}',d'} := HC_{\vec{a}+\vec{a}',d+d'}$. Clearly, for any $l < n^s$ we have

$$(HC_{\vec{a},d} \oplus HC_{\vec{a}',d'})(l) = HC_{\vec{a},d}(l) + HC_{\vec{a}',d'}(l).$$

For each $j = 0, 1, \ldots, r$ let

$\vec{y}_j := HC_{\vec{a}_r,d_r}(0) + \cdots + HC_{\vec{a}_j,d_j}(0) + HC_{\vec{a}_{j-1},d_{j-1}}(m) + \cdots + HC_{\vec{a}_0,d_0}(m)$.

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Then D is a homothetic copy of $[n]^s$.

Claim (2)

The homothetic copy D is monochromatic up to m.

Proof of Claim (2) Note that all elements

$$D(l) := HC_{\vec{a}_r,d_r}(0) + \cdots + HC_{\vec{a}_{j_2},d_{j_2}}(0) \\ + HC_{\vec{a}_{j_2-1}}(l) \oplus \cdots \oplus HC_{\vec{a}_{j_1},d_{j_1}}(l) \\ + HC_{\vec{a}_{j_1-1},d_{j_1-1}}(m) + \cdots + HC_{\vec{a}_0,d_0}(m).$$

for $l = 0, 1, \ldots, m-1$ have the same c–value.

Note also that $D(0) = \vec{y}_{j_1}$ and $D(m) = \vec{y}_{j_2}$ have the same c–value. Hence, the homothetic copy D of $[n]^s$ is monochromatic up to m with respect to c. This completes the proof. □

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The End

Thank you for your attention.