More thoughts on the use of nonstandard methods to extend Roth's Theorem

Steven Leth, University of Northern Colorado

University of Pisa, July, 2023

Roth's Theorem

Let $A \subseteq \mathbb{N}$. The upper Banach density of A is defined by

$$BD(A) = \lim_{n\to\infty} \sup_{k\in\mathbb{N}} \frac{|(A \cap [k, k+n-1])|}{n}.$$

Roth's Theorem (1953) If $A \subseteq \mathbb{N}$ and $BD(A) > 0$, then A contains a 3-term arithmetic progression, i.e. a set of the form $\{a, a+d, a+2d\}$.

This was the first step toward a solution to a question posed by Erdős and Turán, who in 1936 conjectured that every set of positive Banach Density contains arbitrarily long arithmetic progressions. Szemerédi proved the full conjecture in 1975. Since then of course there has been enormous work in combinatorics, by many people, based on Furstenberg's Ergodic Theory proof and Gowers' proof using techniques from Fourier Analysis.

Roth's Theorem

Let $A \subseteq \mathbb{N}$. The upper Banach density of A is defined by

$$BD(A) = \lim_{n\to\infty} \sup_{k\in\mathbb{N}} \frac{|(A \cap [k, k+n-1])|}{n}.$$

Roth's Theorem (1953) If $A \subseteq \mathbb{N}$ and $BD(A) > 0$, then A contains a 3-term arithmetic progression, i.e. a set of the form $\{a, a+d, a+2d\}$.

This was the first step toward a solution to a question posed by Erdős and Turán, who in 1936 conjectured that every set of positive Banach Density contains arbitrarily long arithmetic progressions. Szemerédi proved the full conjecture in 1975. Since then of course there has been enormous work in combinatorics, by many people, based on Furstenberg's Ergodic Theory proof and Gowers' proof using techniques from Fourier Analysis.

Roth's Theorem

Let $A \subseteq \mathbb{N}$. The upper Banach density of A is defined by

$$BD(A) = \lim_{n \to \infty} \sup_{k \in \mathbb{N}} \frac{|(A \cap [k, k+n-1])|}{n}.$$

Roth's Theorem (1953) If $A \subseteq \mathbb{N}$ and $BD(A) > 0$, then A contains a 3-term arithmetic progression, i.e. a set of the form $\{a, a+d, a+2d\}$.

This was the first step toward a solution to a question posed by Erdős and Turán, who in 1936 conjectured that every set of positive Banach Density contains arbitrarily long arithmetic progressions. Szemerédi proved the full conjecture in 1975. Since then of course there has been enormous work in combinatorics, by many people, based on Furstenberg's Ergodic Theory proof and Gowers' proof using techniques from Fourier Analysis.

Lower Bounds

If we were to naively look for subsets of the natural numbers that contain no 3-term arithmetic progressions, the most natural way to construct them might be to start with a small set and then add in successive numbers to the set whenever no 3-term a,p, is introduced. Or, we might try a more "top down" Cantor type approach of removing middle thirds or some similar type of construction. These simple counterexamples would tend to lead to sets of density $\frac{1}{n^c}$, for some constant $c > 0$. Thus if we let $r(n)$ be the maximal density of a subset of $\{1, ..., n\}$ that contains no 3-term arithmetic progression, it is easy to see that for any fixed $c > 0$, and for sufficiently large n

$$r(n) > \frac{1}{n^c}$$

Lower Bounds

This simple lower bound was greatly improved in 1946 by Behrend, using a construction that was based on the fact that any line can intersect a sphere in at most two points. Behrend showed that for sufficiently large n, and a constant $c > 0$ there exist subsets of $\{1, \ldots, n\}$ of density at least $e^{-c(\log n)^{(1/2)}}$ that contain no 3-term arithmetic progression, so that

$$r(n) > e^{-c(\log n)^{(1/2)}}.$$

Although there has been some improvement on the constant over the years, the best known lower bound has not changed significantly since Behrend's result.

Upper Bounds

Roth's Theorem represented the first significant upper bound for $r(n)$, as it says that the cardinality of a the largest subset of $\{1, 2, ...n\}$ that contains no 3-term arithmetic progression must be of order $o(n)$, i.e. for any $\epsilon > 0$ and sufficiently large n, $r(n) < \epsilon$.

The search for better upper bounds has been extensive and in some ways remarkably slow over the years, considering the number of different and powerful techniques that have been developed to address Szemerédi's Theorem and related questions over the years.

Upper Bounds

Roth's Theorem represented the first significant upper bound for $r(n)$, as it says that the cardinality of a the largest subset of $\{1, 2, ...n\}$ that contains no 3-term arithmetic progression must be of order $o(n)$, i.e. for any $\epsilon > 0$ and sufficiently large n, $r(n) < \epsilon$.

The search for better upper bounds has been extensive and in some ways remarkably slow over the years, considering the number of different and powerful techniques that have been developed to address Szemerédi's Theorem and related questions over the years.

Upper Bounds and Conjectures

Erdős (and Turán) further conjectured that any subset A of $\mathbb{N}$ with the property that the sum of the reciprocals of A diverges must contain arbitrarily long arithmetic progressions. This question has, up to this point, only been solved in the 3-term (Roth) case, and only very recently.

Bloom and Sisask have shown, in a paper first posted to the arXiv in 2020 (latest version September 2021), that any set $A \subseteq \mathbb{N}$ that contains no 3-term arithmetic progression has asymptotic density less than $\frac{1}{\log(n)^{1+c}}$ for some positive constant c. So, for sufficiently large n

$$\frac{|A \cap [1, n]|}{n} < \frac{1}{\log(n)^{1+c}}$$

Upper Bounds and Conjectures

Erdős (and Turán) further conjectured that any subset A of $\mathbb{N}$ with the property that the sum of the reciprocals of A diverges must contain arbitrarily long arithmetic progressions. This question has, up to this point, only been solved in the 3-term (Roth) case, and only very recently.

Bloom and Sisask have shown, in a paper first posted to the arXiv in 2020 (latest version September 2021), that any set $A \subseteq \mathbb{N}$ that contains no 3-term arithmetic progression has asymptotic density less than $\frac{1}{\log(n)^{1+c}}$ for some positive constant c. So, for sufficiently large n

$$\frac{|A \cap [1, n]|}{n} < \frac{1}{\log(n)^{1+c}}$$

Conjectures and much stronger upper bounds

Bloom and Sisask indicated in their paper that they believed this result is far from best possible, and conjectured that if we let $r(n)$ be the maximal density of a subset of $\{1, ..., n\}$ that contains no 3-term arithmetic progression, then $r(n)$ is on the order of

$$e^{-c'(\log n)^c}$$

for constants $c, c' > 0$.

Incredibly, this level of result was just recently achieved by Kelley and Meka. Their groundbreaking paper ("Strong Bounds for 3-Progressions") was posted on February 10th of this year, and a revised version just came out on June 18th. Their appendix A in that version contains a nice discussion of how their methods compare to previous work on Roth's Theorem and extensions.

Conjectures and much stronger upper bounds

Bloom and Sisask indicated in their paper that they believed this result is far from best possible, and conjectured that if we let $r(n)$ be the maximal density of a subset of $\{1, ..., n\}$ that contains no 3-term arithmetic progression, then $r(n)$ is on the order of

$$e^{-c'(\log n)^c}$$

for constants $c, c' > 0$.

Incredibly, this level of result was just recently achieved by Kelley and Meka. Their groundbreaking paper ("Strong Bounds for 3-Progressions") was posted on February 10th of this year, and a revised version just came out on June 18th. Their appendix A in that version contains a nice discussion of how their methods compare to previous work on Roth's Theorem and extensions.

Kelley and Meka's result

Kelley and Meka were able to show that

$$r(n) < e^{-c(\log n)^{1/11}}$$

for some constant $c > 0$.

Until Bloom and Sisask's result in 2021 a variety of strong results over the years, using a wide variety of different methods, had succeeded in approaching the logarithmic barrier, i.e. had shown that for sufficiently large n,

$$r(n) < \frac{1}{(\log n)^c}$$

for constants that were getting closer and closer to 1. Bloom and Sisask's result achieved $c > 1$ for the first time. Kelley and Meka's result implies that this is true for all c.

Kelley and Meka's result

Kelley and Meka were able to show that

$$r(n) < e^{-c(\log n)^{1/11}}$$

for some constant $c > 0$.

Until Bloom and Sisask's result in 2021 a variety of strong results over the years, using a wide variety of different methods, had succeeded in approaching the logarithmic barrier, i.e. had shown that for sufficiently large n,

$$r(n) < \frac{1}{(\log n)^c}$$

for constants that were getting closer and closer to 1. Bloom and Sisask's result achieved $c > 1$ for the first time. Kelley and Meka's result implies that this is true for all c.

Jin's Version of Tao's version of a proof of Szemerédi's Theorem

All currently known proofs of Szemerédi's Theorem are difficult in their own way. Szemerédi's original proof is generally considered a masterpiece of combinatorial reasoning, but is extremely difficult to fully understand. Tao rewrote the proof several times, and in 2017 asked if nonstandard methods could make the proof simpler and more accessible. In 2022 Jin succeeded in using a nonstandard framework to simplify and clarify Tao's version of Szemerédi's proof, and discussed his proof at "UltraMath 2022" here in Pisa.

Armed with a much better understanding of Szemerédi's Theorem due to Jin's proof, I was interested in the question of whether or not the combinatorial proof in the nonstandard setting could provide a pathway to proofs for the cases of smaller density.

Jin's Version of Tao's version of a proof of Szemerédi's Theorem

All currently known proofs of Szemerédi's Theorem are difficult in their own way. Szemerédi's original proof is generally considered a masterpiece of combinatorial reasoning, but is extremely difficult to fully understand. Tao rewrote the proof several times, and in 2017 asked if nonstandard methods could make the proof simpler and more accessible. In 2022 Jin succeeded in using a nonstandard framework to simplify and clarify Tao's version of Szemerédi's proof, and discussed his proof at "UltraMath 2022" here in Pisa.

Armed with a much better understanding of Szemerédi's Theorem due to Jin's proof, I was interested in the question of whether or not the combinatorial proof in the nonstandard setting could provide a pathway to proofs for the cases of smaller density.

The framework for the Tao/Jin Roth proof

We assume for the sake of contradiction that Roth's Theorem is not true, so that there exists a minimal $\alpha > 0$ such that any sufficiently large set of density greater than α contains a 3-term a.p., but that for every $\epsilon > 0$ there exist arbitrarily large finite n, and sets A_n, such that A_n contains no 3-term a.p. and

$$A_n \cap [0, n] > (\alpha - \epsilon)n.$$

For any internal set $B \subseteq N$ and any non-finite H, we will write

$$\delta_H(B) := \frac{|B|}{H}$$

and $\mu_H(B)$ for the Loeb measure of B with respect to H. We note that $\delta_H(B)$ is an internal quantity and $\mu_H(B)$ is external.

The framework for the Tao/Jin Roth proof

We assume for the sake of contradiction that Roth's Theorem is not true, so that there exists a minimal $\alpha > 0$ such that any sufficiently large set of density greater than α contains a 3-term a.p., but that for every $\epsilon > 0$ there exist arbitrarily large finite n, and sets A_n, such that A_n contains no 3-term a.p. and

$$A_n \cap [0, n] > (\alpha - \epsilon)n.$$

For any internal set $B \subseteq N$ and any non-finite H, we will write

$$\delta_H(B) := \frac{|B|}{H}$$

and $\mu_H(B)$ for the Loeb measure of B with respect to H. We note that $\delta_H(B)$ is an internal quantity and $\mu_H(B)$ is external.

super uniformity of density

We let $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, and $A \subseteq [1, N] = [N]$ be a maximal counterexample to Roth's Theorem. Then

$$\delta_N(A) \approx \alpha \text{ and } \mu_N(A) = \alpha,$$

and this maximal counterexample has a sort of "super uniformity" property of densities:

- Any arithmetic progression P of non-finite length in $[N]$ must satisfy $\mu_{I(P)}(A \cap P) \leq \alpha$.
- Any collection of arithmetic progressions $\{P_i \subseteq [N] : i \in [I]\}$ of non-finite length L such that $\mu_N(I) > 0$ must satisfy $\mu_L(A \cap P_i) = \alpha$ for (μ_I)-almost all $i \in [I]$.
- For any H non-infinitesimal compared to N, and for all $x \in [N - H]$

$$\mu_H(A \cap (x + [H])) = \alpha$$

Slight Variation of the Tao/Jin Proof for Roth

Let E_0 be the set of all points in $[0, H]$ that are halfway between the point 0 and an element of A. Equivalently, if we let A_e be the even elements in $A \cap [0, 2H]$, then $E_0 = \frac{1}{2}A_e$.

It is easy to see that E_0 has the same "super uniformity" properties as A, and is also a maximally dense set with no 3-term arithmetic progressions with $\mu_H(E_0) = \alpha$

Although stated in a different form, Tao and Jin's proofs of Roth's Theorem use a version of the Regularity Lemma to show that almost all elements of $[0, H]$ are in $A - E_0$.

This means that $\mu((A - E_0) \cap E_0) = \alpha$, so that there exists a non-trivial intersection of $A - E_0$ and E_0, i.e. an $e_1 \neq e_2$ and an $a \in A$ such that $a = e_1 + e_2$. It is easy to see that this implies that $\{2e_1, a, 2e_2\}$ is a 3-term arithmetic progression.

Slight Variation of the Tao/Jin Proof for Roth

Let E_0 be the set of all points in $[0, H]$ that are halfway between the point 0 and an element of A. Equivalently, if we let A_e be the even elements in $A \cap [0, 2H]$, then $E_0 = \frac{1}{2} A_e$.

It is easy to see that E_0 has the same “super uniformity” properties as A, and is also a maximally dense set with no 3-term arithmetic progressions with $\mu_H(E_0) = \alpha$

Although stated in a different form, Tao and Jin's proofs of Roth's Theorem use a version of the Regularity Lemma to show that almost all elements of $[0, H]$ are in $A - E_0$.

This means that $\mu((A - E_0) \cap E_0) = \alpha$, so that there exists a non-trivial intersection of $A - E_0$ and E_0, i.e. an $e_1 \neq e_2$ and an $a \in A$ such that $a = e_1 + e_2$. It is easy to see that this implies that $\{2e_1, a, 2e_2\}$ is a 3-term arithmetic progression.

Slight Variation of the Tao/Jin Proof for Roth

Let E_0 be the set of all points in $[0, H]$ that are halfway between the point 0 and an element of A. Equivalently, if we let A_e be the even elements in $A \cap [0, 2H]$, then $E_0 = \frac{1}{2}A_e$.

It is easy to see that E_0 has the same "super uniformity" properties as A, and is also a maximally dense set with no 3-term arithmetic progressions with $\mu_H(E_0) = \alpha$

Although stated in a different form, Tao and Jin's proofs of Roth's Theorem use a version of the Regularity Lemma to show that almost all elements of $[0, H]$ are in $A - E_0$.

This means that $\mu((A - E_0) \cap E_0) = \alpha$, so that there exists a non-trivial intersection of $A - E_0$ and E_0, i.e. an $e_1 \neq e_2$ and an $a \in A$ such that $a = e_1 + e_2$. It is easy to see that this implies that $\{2e_1, a, 2e_2\}$ is a 3-term arithmetic progression.

Slight Variation of the Tao/Jin Proof for Roth

Let E_0 be the set of all points in $[0, H]$ that are halfway between the point 0 and an element of A. Equivalently, if we let A_e be the even elements in $A \cap [0, 2H]$, then $E_0 = \frac{1}{2}A_e$.

It is easy to see that E_0 has the same "super uniformity" properties as A, and is also a maximally dense set with no 3-term arithmetic progressions with $\mu_H(E_0) = \alpha$

Although stated in a different form, Tao and Jin's proofs of Roth's Theorem use a version of the Regularity Lemma to show that almost all elements of $[0, H]$ are in $A - E_0$.

This means that $\mu((A - E_0) \cap E_0) = \alpha$, so that there exists a non-trivial intersection of $A - E_0$ and E_0, i.e. an $e_1 \neq e_2$ and an $a \in A$ such that $a = e_1 + e_2$. It is easy to see that this implies that $\{2e_1, a, 2e_2\}$ is a 3-term arithmetic progression.

A version of Szemerédi's Regularity Lemma

The following version of the Weak Regularity Lemma is used in their proofs:

Weak Regularity Lemma Let V, W be finite sets, let $\epsilon > 0$, and for each $w \in W$, let E_w be a subset of V. Then there exists a partition $V = V_1 \cup V_2 \cup \ldots \cup V_n$ with $n = O(b^{1/\epsilon})$ for some standard real $b > 1$, and real numbers $0 \leq c_{i,w} \leq 1$ for $i \leq n$ and $w \in W$ such that for any set $\subseteq V$, one has

$$\left| |F \cap E_w| - \sum_{i=1}^{n} c_{i,w} |F \cap V_i| \right| \leq \epsilon |V|$$

for all but $\epsilon |W|$ values of $w \in W$.

If we want to use this approach on sets of density $\frac{1}{\log n}$ or less, then we can only choose epsilons that are roughly the same size as our densities, which is not sufficient for obtaining useful results.

A version of Szemerédi's Regularity Lemma

The following version of the Weak Regularity Lemma is used in their proofs:

Weak Regularity Lemma Let V, W be finite sets, let $\epsilon > 0$, and for each $w \in W$, let E_w be a subset of V. Then there exists a partition $V = V_1 \cup V_2 \cup \ldots \cup V_n$ with $n = O(b^{1/\epsilon})$ for some standard real $b > 1$, and real numbers $0 \leq c_{i,w} \leq 1$ for $i \leq n$ and $w \in W$ such that for any set $\subseteq V$, one has

$$\left| |F \cap E_w| - \sum_{i=1}^{n} c_{i,w} |F \cap V_i| \right| \leq \epsilon |V|$$

for all but $\epsilon|W|$ values of $w \in W$.

If we want to use this approach on sets of density $\frac{1}{\log n}$ or less, then we can only choose epsilons that are roughly the same size as our densities, which is not sufficient for obtaining useful results.

Moving toward extensions

With this in mind I have continued to think about ways to prove Roth's Theorem using this nonstandard approach, while only using tools that have a reasonable chance of working for smaller density sets. I have not yet been successful in this regard, but I would like to outline some of the approaches that look somewhat promising.

Ideally we will be able to prove that $A - E_0$ contains almost all of $[0, H]$ without the Regularity Lemma. A proof that succeeds in this regard would be a very good model for sets of lower density, but this is more than is needed to complete the proof of Roth's Theorem. Since both A and E_0 have the "super uniformity" density condition, it is sufficient to prove that the set of points that are contained in arithmetic progressions of non-finite length in $A - E_0$ or in $E_0 + E_0$ is of positive measure and the differences of the progression are the same. I think this is true even if the differences are not the same, but the proof is not yet clear to me.

Moving toward extensions

With this in mind I have continued to think about ways to prove Roth's Theorem using this nonstandard approach, while only using tools that have a reasonable chance of working for smaller density sets. I have not yet been successful in this regard, but I would like to outline some of the approaches that look somewhat promising.

Ideally we will be able to prove that $A - E_0$ contains almost all of $[0, H]$ without the Regularity Lemma. A proof that succeeds in this regard would be a very good model for sets of lower density, but this is more than is needed to complete the proof of Roth's Theorem. Since both A and E_0 have the "super uniformity" density condition, it is sufficient to prove that the set of points that are contained in arithmetic progressions of non-finite length in $A - E_0$ or in $E_0 + E_0$ is of positive measure and the differences of the progression are the same. I think this is true even if the differences are not the same, but the proof is not yet clear to me.

A starting Lemma

Lemma

Let $1 \ll N$, $A \subseteq [N]$ *be a maximal counterexample to Roth's theorem as before,* $1 \ll H < N/2$, *and* $R \subseteq [N - H]$ *be an a.p. with* $|R| \gg 1$.
Then:
(i) *For a set* $E \subseteq H$ *with* $\mu_H(E) > 0$, *and for any a.p.* $P \subseteq R$ *of non-finite length, there is an* $x \in P$ *such that*

$$\mu_H(A \cap (x + E)) \geq \alpha \mu_H(E)$$

(ii) *Let* $E \subseteq H$ *with* $\mu_H(E) > 0$. *Then for any non-infinitesimal* $\epsilon > 0$, *and any a.p.* $P \subseteq R$ *of non-finite length, there exists an interval* $I \subseteq P$ *of non-finite length such that for all* x *in* I,

$$\delta_H(A \cap (x + E)) \geq (1 - \epsilon)\alpha \delta_H(E).$$

Proof Sketch

The proof of part (i) is the same double counting argument used by Jin (this form of the statement is taken directly from that proof). Intuitively, for a fixed element of E, as we move through the x's in P, we will almost always intersect with α many elements in A, and will never intersect with more than that. If no x satisfies the conclusion we get a contradiction by double counting $\alpha\mu_H(E)$.

We note that part (i) immediately implies that there exists some finite m such that every interval of length m in any arithmetic progression contains an element of $A - E$, in fact an element that is the difference of at least $(1-\epsilon)\alpha\delta_H(E)$ different elements in A (and corresponding elements in E).

Proof Sketch

The proof of part (i) is the same double counting argument used by Jin (this form of the statement is taken directly from that proof). Intuitively, for a fixed element of E, as we move through the x's in P, we will almost always intersect with α many elements in A, and will never intersect with more than that. If no x satisfies the conclusion we get a contradiction by double counting $\alpha\mu_H(E)$.

We note that part (i) immediately implies that there exists some finite m such that every interval of length m in any arithmetic progression contains an element of $A - E$, in fact an element that is the difference of at least $(1-\epsilon)\alpha\delta_H(E)$ different elements in A (and corresponding elements in E).

Proof Sketch continued

If the conclusion for part (*ii*) is not true, then for some finite m, every interval of length m in P contains an x such that

$$\delta_H(A \cap (x + E_w)) < (1 - \epsilon)\alpha\delta_H(E_w).$$

We now "color" the blocks of length m in E_w in m colors based on where the first such x in each block occurs.

By van Der Waerden's Theorem, there exist arbitrarily long blocks in which there is such a "bad" x in the same position. By "overspill" there exists an infinitely long progression of such blocks in P, and therefore there exists an infinitely long progression of such x themselves. This contradicts part (i) with $E = E_w$.

Proof Sketch continued

If the conclusion for part (*ii*) is not true, then for some finite m, every interval of length m in P contains an x such that

$$\delta_H(A \cap (x + E_w)) < (1 - \epsilon)\alpha\delta_H(E_w).$$

We now "color" the blocks of length m in E_w in m colors based on where the first such x in each block occurs.

By van Der Waerden's Theorem, there exist arbitrarily long blocks in which there is such a "bad" x in the same position. By "overspill" there exists an infinitely long progression of such blocks in P, and therefore there exists an infinitely long progression of such x themselves. This contradicts part (i) with $E = E_w$.

An observation

If we apply the Lemma to $E = E_0$, it is problematic that the densities of the $A \cap (x + E_0))$ sets are not uniform in measure (or not easily seen to be somewhat uniform without the Regularity Lemma). For example, if we knew that the measures never get much above α^2 then there could not be many x for which the measure is zero, and $A - E_0$ would be large.

Interestingly, the Lemma shows us that the set of x's for which the measure is infinitesimally close to α^2 is "large" in some "van Der Waerden sense."

If we apply the Lemma to $E = E_0$, it is problematic that the densities of the $A \cap (x + E_0))$ sets are not uniform in measure (or not easily seen to be somewhat uniform without the Regularity Lemma). For example, if we knew that the measures never get much above α^2 then there could not be many x for which the measure is zero, and $A - E_0$ would be large.

Interestingly, the Lemma shows us that the set of x's for which the measure is infinitesimally close to α^2 is "large" in some "van Der Waerden sense."

An observation

More specifically, if we were to pick any standard $\epsilon > 0$ and three-color the x values based on those that have

$$1) \qquad \delta_H(A \cap (x + E_0)) < \alpha^2 - \epsilon$$

$$2) \qquad \alpha^2 - \epsilon < \delta_H(A \cap (x + E_0)) < \alpha^2 + \epsilon$$

$$3) \qquad \delta_H(A \cap (x + E_0)) > \alpha^2 - \epsilon$$

Then by the lemma and the super uniformity condition, there must be arbitrarily long arithmetic progressions for color 2 (and none for the other colors).

Many long progressions in $A - E_0$

Define

$$G := \{x \in H : \delta_H(A \cap (x + E_0)\} \geq (1-\epsilon)\alpha^2$$

Note that $G \subset A - E_0$. More specifically, G is the set of all elements of $A - E_0$ that are the difference of at least $(1-\epsilon)\alpha^2 H$ many points in A and corresponding points in E_0.

From the lemma we know that G is not only of positive Loeb measure, but is, in fact syndetic. Thus, the next result seems to be an important step toward a proof of Roth's Theorem. Without the Regularity Lemma it is very difficult to control both the location and the size of the differences in progressions, but the next result shows that every element of G is an a long arithmetic progression.

Many long progressions in $A - E_0$

Define

$$G := \{x \in H : \delta_H(A \cap (x + E_0)\} \geq (1 - \epsilon)\alpha^2$$

Note that $G \subset A - E_0$. More specifically, G is the set of all elements of $A - E_0$ that are the difference of at least $(1 - \epsilon)\alpha^2 H$ many points in A and corresponding points in E_0.

From the lemma we know that G is not only of positive Loeb measure, but is, in fact syndetic. Thus, the next result seems to be an important step toward a proof of Roth's Theorem. Without the Regularity Lemma it is very difficult to control both the location and the size of the differences in progressions, but the next result shows that every element of G is an a long arithmetic progression.

Many long progressions in $A - E_0$

Proposition

Every element $u \in G$ is an initial point of an arithmetic progression of infinite length in G.

Proof

Let u be any element in G, and consider the arithmetic progressions P_n that start at u and increase by multiples of n. If the elements of these progressions are in G for arbitrarily long initial segments, then the result follows by overspill, since in that case there must be some N for which the first element not in $G \cap P_N$ is a non-finite multiple of N.

Proof of the Proposition, continued

If not, then for some finite m, one of the first m elements in each P_n is not in G. We color the natural numbers n in m colors based on which element is the first in P_n to not be in G.

We can then obtain arbitrarily long arithmetic progressions corresponding to some fixed number $k \leq m$. Then k times this progression of n values is an arbitrarily long a.p. of elements not in G, contradicting the lemma. □

Proof of the Proposition, continued

If not, then for some finite m, one of the first m elements in each P_n is not in G. We color the natural numbers n in m colors based on which element is the first in P_n to not be in G.

We can then obtain arbitrarily long arithmetic progressions corresponding to some fixed number $k \leq m$. Then k times this progression of n values is an arbitrarily long a.p. of elements not in G, contradicting the lemma. $\square$

Translates/Shifts

It is now helpful to insert many "shifts," with the goal of packing the progressions closer together (even at the cost of shortening them considerably). The next lemma is one way to start on this goal.

For simplicity the finite number m is playing a dual role in the lemma below, although this could be confusing. The point is that the set of all x's for which $A \cap (x + A) - E_0$ contains no gaps greater than m is itself a set with no gap of size greater than m.

Translates/Shifts

It is now helpful to insert many "shifts," with the goal of packing the progressions closer together (even at the cost of shortening them considerably). The next lemma is one way to start on this goal.

For simplicity the finite number m is playing a dual role in the lemma below, although this could be confusing. The point is that the set of all x's for which $A \cap (x + A) - E_0$ contains no gaps greater than m is itself a set with no gap of size greater than m.

Proof

Proof By part (*ii*) of Lemma 1, there exists an m_1 such that every interval of length m_1 contains an x such that

$$\delta_H(A \cap (x + a)) \geq (1 - \epsilon)\alpha^2.$$

By the super-uniformity property of E_0, there exists an m_2 such that the collection of intervals of length m_2 that intersect E_0 has measure greater than $1 - \alpha^2$. Then for any k there exists j such that both

$$[(j+k)m_2, (j+k)m_2 + m_2)] \cap (A \cap (x + A) \text{ and } [jm_2, jm_2 + m_2)] \cap E_0$$

are nonempty. This means that there is an element of $A \cap (x + A) - E_0$ in the interval $[km_2, km_2 + m_2]$. We may now let m be the larger of m_1 and m_2. □

It might be better to use the fact that that we can continue to translate by arbitrarily many finite values. Since both A and E_0 satisfy the super-uniformity conditions, we may continue to insert various "translates" into the situation. For example, for arbitrarily long sequences of x_i's in $[0, H]$,

$$\mu(x_1 + A) \cap (x_2 + E_0) \cap (x_3 + A) \cap \ldots \cap (x_{2n} + E_0) \geq \alpha^{2n}.$$

Of course, since the densities are decreasing there is no longer a fixed m as in the previous result. So, we cannot make sure that these values are close together. Nevertheless, it seems that a large measure of long arithmetic progressions, together with the ability to translate them, or parts of them, should allow us to get a positive measure of long progressions of one fixed difference, which is sufficient to prove the result.

Thank you to the organizers of this lovely event!!!!

and to our friend Mauro......

Salute!