NONSTANDARD METHODS WITHOUT THE AXIOM OF CHOICE

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PISA, July 11-12, 2023

INTRODUCTION.

Nonstandard methods have been successfully applied to standard problems in finite combinatorics and number theory by Renling Jin, Terence Tao, Mauro DiNasso and many others.

Nonstandard Analysis is sometimes criticized for its implicit dependence on the Axiom of Choice (**AC**).
(Bishop, Connes,..)

Indeed, model-theoretic frameworks for nonstandard methods entail the existence of nonprincipal ultrafilters over $\mathbb{N}$, a strong form of **AC**:

If $*$ is the mapping that assigns to each $X \subseteq \mathbb{N}$ its nonstandard extension ${}^*X$, and if $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$ is an unlimited integer, then the set $U = \{X \subseteq \mathbb{N} \mid \nu \in {}^*X\}$ is a nonprincipal ultrafilter over $\mathbb{N}$.

The common axiomatic/syntactic frameworks for nonstandard methods, such as **IST** or **HST**, include **ZFC** among their axioms. The dependence on **AC** cannot be avoided by simply removing it from the list of axioms. These theories postulate some version of the *Standardization Principle*, according to which *for every formula* Φ *in the language of the theory (possibly with parameters) and every standard set* A *there exists a standard set* S *such that for all standard* x,

$$x \in S \iff x \in A \wedge \Phi(x).$$

This set is denoted ${}^{\mathrm{st}}\{x \in A \mid \Phi(x)\}$.

It follows that, for an unlimited $\nu \in \mathbb{N}$, the standard set $U = {}^{\mathrm{st}}\{X \in \mathcal{P}(\mathbb{N}) \mid \nu \in X\}$ is a nonprincipal ultrafilter over $\mathbb{N}$.

In this sense, all results obtained in Nonstandard Analysis depend on the Axiom of Choice.

While strong forms of **AC**, such as Zorn's Lemma, are instrumental in many abstract areas of mathematics, such as general topology (the product of compact spaces is compact), measure theory (there exist sets that are not Lebesgue measurable) or functional analysis (Hahn-Banach theorem), it is undesirable to have to rely on them for results in "ordinary" mathematics such as calculus, finite combinatorics and number theory.

SPOT

In the paper
Mikhail G. Katz and KH,
Infinitesimal analysis without the Axiom of Choice,
Ann. Pure Appl. Logic 172, 6 (2021)
https://doi.org/10.1016/j.apal.2021.102959
https://arxiv.org/abs/2009.04980

we have formulated a set theory **SPOT** in the st-$\in$-language.

The theory has three simple axioms, Standard Part, nOntriviality and Transfer.
It is a subtheory of the nonstandard set theories **IST** and **HST**, but unlike them, it is a conservative extension of **ZF.** Arguments carried out in **SPOT** thus do not depend on any form of **AC.**

By an $\in$-language we mean the language that contains a binary membership predicate $\in$ and is enriched by defined symbols for constants, relations, functions and operations customary in traditional mathematics.

For example, it contains names $\mathbb{N}$ and $\mathbb{R}$ for the sets of natural and real numbers; they are viewed as defined in the traditional way ($\mathbb{N}$ is the least inductive set, $\mathbb{R}$ is defined in terms of Dedekind cuts or Cauchy sequences).

The classical theories **ZF** and **ZFC** are formulated in the $\in$-language.

The language of **SPOT** contains an additional unary predicate st, where $\mathsf{st}(x)$ reads "x is standard."

We use $\forall$ and $\exists$ as quantifiers over sets and $\forall^{\mathsf{st}}$ and $\exists^{\mathsf{st}}$ as quantifiers over standard sets.

The "nonstandard" axioms of **SPOT** reflect the insights of Leibniz.

They are:

ZF (Zermelo - Fraenkel Set Theory)

T (Transfer) Let ϕ be an $\in$-formula with standard parameters. Then

$$\forall^{\text{st}} x \; \phi(x) \Rightarrow \forall x \; \phi(x).$$

O (Nontriviality) $\quad \exists \nu \in \mathbb{N} \; \forall^{\text{st}} n \in \mathbb{N} \; (n \neq \nu).$

SP (Standard Part)

$$\forall A \subseteq \mathbb{N} \; \exists^{\text{st}} B \subseteq \mathbb{N} \; \forall^{\text{st}} n \in \mathbb{N} \; (n \in B \; \leftrightarrow \; n \in A).$$

Some of the general results provable in **SPOT** are:

Proposition. *Standard natural numbers precede all nonstandard ones:* $\forall^{\mathrm{st}} n \in \mathbb{N}\, \forall m \in \mathbb{N}\, (m < n \Rightarrow \mathrm{st}(m))$.

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Proposition. (Countable Idealization)
Let ϕ be an $\in$-formula with arbitrary parameters.

$$\forall^{\mathrm{st}} n \in \mathbb{N}\ \exists x\ \forall m \in \mathbb{N}\ (m \leq n\ \Rightarrow \phi(m, x))\ \leftrightarrow\ \exists x\ \forall^{\mathrm{st}} n \in \mathbb{N}\ \phi(n, x).$$

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Countable Idealization easily implies the following more familiar form. We use $\forall^{\mathrm{st\,fin}}$ and $\exists^{\mathrm{st\,fin}}$ as quantifiers over standard finite sets.
Let ϕ be an $\in$-formula with arbitrary parameters. For every standard countable set A

$$\forall^{\mathrm{st\,fin}} a \subseteq A\, \exists x\, \forall y \in a\, \phi(x, y) \;\leftrightarrow\; \exists x\, \forall^{\mathrm{st}} y \in A\, \phi(x, y).$$

The axiom **SP** is often used in the equivalent form

$$\forall x \in \mathbb{R}\ (x \text{ limited } \Rightarrow \exists^{\mathrm{st}} r \in \mathbb{R}\ (x \simeq r)) \qquad (\mathbf{SP}')$$

where x is *limited* iff $|x| \leq n$ for some standard $n \in \mathbb{N}$, and $x \simeq r$ iff $|x - r| \leq 1/n$ for all standard $n \in \mathbb{N}$, $n \neq 0$; x is *infinitesimal* if $x \simeq 0 \wedge x \neq 0$.
The unique standard real number r is called the *standard part of x* or the *shadow of x*; notation $r = \mathrm{sh}(x)$.

The theory **SPOT** proves two important stronger versions of **SP**.

Definition. An st-$\in$-formula $\Phi(v_1, \ldots, v_r)$ is *special* if it is of the form

$$\mathrm{Q}^{\mathrm{st}} u_1 \ldots \mathrm{Q}^{\mathrm{st}} u_s \, \psi(u_1, \ldots, u_s, v_1, \ldots, v_r)$$

where ψ is an $\in$-formula and each Q stands for $\exists$ or $\forall$.

We use $\forall^{\mathrm{st}}_{\mathbb{N}} u \ldots$ and $\exists^{\mathrm{st}}_{\mathbb{N}} u \ldots$ as shorthand for respectively $\forall^{\mathrm{st}} u \, (u \in \mathbb{N} \Rightarrow \ldots)$ and $\exists^{\mathrm{st}} u \, (u \in \mathbb{N} \wedge \ldots)$.
An $\mathbb{N}$-*special* formula is a formula of the form

$$\mathrm{Q}^{\mathrm{st}}_{\mathbb{N}} u_1 \ldots \mathrm{Q}^{\mathrm{st}}_{\mathbb{N}} u_s \, \psi(u_1, \ldots u_s, v_1, \ldots, v_r)$$

where ψ is an $\in$-formula.

Proposition. (Countable Standardization for $\mathbb{N}$-Special Formulas)
Let Φ be an $\mathbb{N}$-special formula with arbitrary parameters. Then

$$\exists^{\mathrm{st}} S\, \forall^{\mathrm{st}} n\, (n \in S \;\leftrightarrow\; n \in \mathbb{N} \wedge \Phi(n)).$$

Of course, $\mathbb{N}$ can be replaced by any standard countable set.

The second version involves special formulas with only the standard parameters.

Proposition. *Let* $\Phi(v_1,\ldots,v_r)$ *be a special formula with standard parameters. Then* $\forall^{\mathrm{st}} A\, \exists^{\mathrm{st}} S\, \forall^{\mathrm{st}} x_1,\ldots,x_r$ $\langle x_1,\ldots,x_r\rangle \in S \;\leftrightarrow\; \langle x_1,\ldots,x_r\rangle \in A \;\wedge\; \Phi(x_1,\ldots,x_r)$.

MATHEMATICS IN SPOT.

Infinitesimal calculus can be developed in **SPOT** as far as the global version of Peano's Theorem.

Theorem.

Let $F : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be a continuous function. There is an interval $[0,a)$ with $0 < a \leq \infty$ and a function $y : [0,a) \to \mathbb{R}$ such that

$$y(0) = 0, \qquad y'(x) = F(x, y(x))$$

holds for all $x \in [0,a)$, and if $a \in \mathbb{R}$ then $\lim_{x \Rightarrow a^-} y(x) = \pm\infty$.

See
M. G. Katz and KH, *Peano and Osgood theorems via effective infinitesimals*, to appear.

The usual proofs of the global version of Peano Theorem use Zorn's Lemma or **ADC** (the Axiom of Dependent Choice).

SPOT proves the existence of densities as defined by Renling Jin.

Strong Upper Banach Densities. In our notation:
For finite $A \subseteq \mathbb{N}$ with $|A|$ unlimited, the *strong upper Banach density of A* is defined by

$$\mathrm{SD}(A) = \sup{}^{\mathrm{st}}\{\mathrm{sh}(|A \cap P|/|P|) \mid |P| \text{ is unlimited}\}.$$

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If $S \subseteq \mathbb{N}$ has $\mathsf{SD}(S) = \eta \in \mathbb{R}$ (note η is standard) and $A \subseteq S$, the *strong upper Banach density of A relative to S* is defined by $\mathsf{SD}_S(A) =$

$$\sup{}^{\mathsf{st}}\{\mathsf{sh}(|A \cap P|/|P|) \mid |P| \text{ is unlimited} \wedge \mathsf{sh}(|S \cap P|/|P|) = \eta\}.$$

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$\mathrm{SD}_S(A) = \sup^{\mathrm{st}}\{q \in \mathbb{Q} \mid \Phi(q)\}$ where $\Phi(q)$ is the formula

$\exists P\,[\forall^{st}_{\mathbb{N}} i\,(|P| > i) \wedge \forall^{st}_{\mathbb{N}} j(|\,|S \cap P|/|P| - \eta| < \frac{1}{j+1}) \wedge q \leq |A \cap P|/|P|\,]$.

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which, upon the exchange of the order of $\exists P$ and $\forall^{st}_{\mathbb{N}} i$, enabled by Countable Idealization, converts to a special st-$\in$-formula

$$\forall^{st}_{\mathbb{N}} i\, \exists P\, [\,(|P| > i) \wedge (|\,|S \cap P|/|P| - \eta| < \tfrac{1}{i+1}) \wedge q \leq |A \cap P|/|P|\,],$$

SCOT

Traditional proofs in “ordinary” mathematics either do not use **AC** at all, or refer only to its weak forms, notably the Axiom of Countable Choice (**ACC**) or the stronger Axiom of Dependent Choice (**ADC**). These axioms are generally accepted and often used without comment.

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These weak forms are necessary to prove eg. the equivalence of the ε-δ definition and the sequential definition of continuity for functions $f : \mathbb{R} \to \mathbb{R}$, or the countable additivity of Loeb measure, but they do not imply the strong consequences of **AC** such as the existence of nonprincipal ultrafilters or the Banach–Tarski paradox.

The theory **SCOT** is **SPOT** + **SN** + **DC**, where

SN (Standardization for st-$\in$-formulas with no parameters or, equivalently, with standard parameters):
Let $\Phi(v)$ *be an* st-$\in$*-formula with no parameters. Then*

$$\forall^{\text{st}} A\, \exists^{\text{st}} S\, \forall^{\text{st}} x\, (x \in S \;\leftrightarrow\; x \in A \wedge \Phi(x)).$$

DC (Dependent Choice for st-$\in$-formulas):
Let $\Phi(u, v)$ *be an* st-$\in$*-formula with arbitrary parameters.*
If B *is a set,* $b \in B$ *and* $\forall x \in B\, \exists y \in B\, \Phi(x, y)$*, then there is a sequence* $\langle b_n \mid n \in \mathbb{N}\rangle$ *such that* $b_0 = b$ *and*
$\forall^{\text{st}} n \in \mathbb{N}\, (b_n \in B \wedge \Phi(b_n, b_{n+1}))$.

Some general consequences of **SCOT**:

CC (Countable st-$\in$-Choice)
Let $\Phi(u, v)$ *be an* st-$\in$*-formula with arbitrary parameters. Then*
$\forall^{\mathrm{st}} n \in \mathbb{N}\, \exists x\, \Phi(n, x) \Rightarrow \exists f\, (f \text{ is a function } \wedge \forall^{\mathrm{st}} n \in \mathbb{N}\, \Phi(n, f(n)))$.

SC (Countable Standardization)
Let $\Psi(v)$ *be an* st-$\in$*-formula with arbitrary parameters. Then*

$$\exists^{\mathrm{st}} S\, \forall^{\mathrm{st}} n\, (n \in S \;\leftrightarrow\; n \in \mathbb{N} \wedge \Psi(n)).$$

SCOT is a conservative extension of **ZF** $+$ **ADC**.

It allows such features as an infinitesimal construction of the Lebesgue measure.

It implies the axioms of Nelson's *Radically Elementary Probability Theory*.

THEORIES WITH MANY LEVELS OF STANDARDNESS

Such theories have been developed in
Y. Péraire, *Théorie relative des ensembles internes*, Osaka J. Math. 29 (1992), 267–297 (**RIST**)

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KH, *Relative set theory: Internal view,* Journal of Logic and Analysis 1:8 (2009), 1–108. doi: 10.4115/jla.2009.1.8 (**GRIST**).

In the book KH, O. Lessmann and R. O'Donovan, *Analysis using Relative Infinitesimals*, Chapman and Hall, 2015, 316 pp. a simple subtheory of these theories is used to develop elementary calculus.

Renling Jin recently gave a groundbreaking nonstandard proof of Szemerédi's theorem:
If D has a positive upper density, then D contains a k-term arithmetic progression for every $k \in \mathbb{N}$
in a model-theoretic framework that has three levels of infinity.

R. Jin, *A simple combinatorial proof of Szemerédi's theorem via three levels of infinities*
`https://arXiv.org/abs/2203.06322v1`

There are four universes ($\mathbb{V}_0, \mathbb{V}_1, \mathbb{V}_2$ and $\mathbb{V}_3$) and some additional elementary embeddings.
Let $\mathbb{N}_j = \mathbb{N} \cap \mathbb{V}_j$ and $\mathbb{R}_j = \mathbb{R} \cap \mathbb{V}_j$ for $j = 0, 1, 2, 3$.

Jin's *Property 2.1* summarizes what is required.
We restate it in a form suitable for axiomatic treatment.

0. $\mathbb{V}_0 \prec \mathbb{V}_1 \prec \mathbb{V}_2 \prec \mathbb{V}_3$.
1. $\mathbb{N}_{j+1}$ is an end extension of $\mathbb{N}_j$ ($j = 0, 1, 2$).
2. For $j' > j$, Countable Idealization holds from $\mathbb{V}_j$ to $\mathbb{V}_j'$:
Let ϕ be an $\in$-formula with parameters from $\mathbb{V}_{j'}$.
$\forall n \in \mathbb{N}_j \, \exists x \, \forall m \in \mathbb{N} \, (m \le n \Rightarrow \phi(m, x)) \leftrightarrow \exists x \, \forall n \in \mathbb{N}_j \, \phi(n, x)$.
3. There is an elementary embedding i_* of $(\mathbb{V}_2; \mathbb{R}_0, \mathbb{R}_1)$ to $(\mathbb{V}_3; \mathbb{R}_1, \mathbb{R}_2)$.
4. There is an elementary embedding i_1 of $\mathbb{V}_1$ to $\mathbb{V}_2$ such that $i_1 \restriction \mathbb{N}_0$ is an identity map and $i_1(a) \in \mathbb{N}_2 \setminus \mathbb{N}_1$ for each $a \in \mathbb{N}_1 \setminus \mathbb{N}_0$.
5. There is an elementary embedding i_2 of $\mathbb{V}_2$ to $\mathbb{V}_3$ such that $i_2 \restriction \mathbb{N}_1$ is an identity map and $i_2(a) \in \mathbb{N}_3 \setminus \mathbb{N}_2$ for each $a \in \mathbb{N}_2 \setminus \mathbb{N}_1$.

We consider extensions of **SPOT** and **SCOT** to many levels of standardness that interpret Property 2.1.

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The language $\mathcal{L}_\infty$ has a binary predicate symbol $\in$, a unary predicate symbol st_S (*S-standard*) for every finite set S of natural numbers, and a unary function symbol $\mathbf{i}_S^{S'}$ for all finite S, S' of the same cardinality (standing for an isomorphism between the universes of S-standard and S'-standard sets).

Note that $\mathsf{st}_n = \mathsf{st}_{\{0,\ldots,n-1\}}$.
We write st for $\mathsf{st}_\emptyset$.
We also let $\mathbb{S}_S = \{x \mid \mathsf{st}_S(x)\}$ and $\mathbb{I}_S^{S'} = \{\langle x, y\rangle \mid \mathbf{i}_S^{S'}(x) = y\}$.

A formula Φ is *simple* if no $\mathbf{i}_S^{S'}$ occur in it, each quantifier is restricted to $\mathtt{st}_n$ for some natural number n, and there are no other occurrences of $\mathtt{st}_S$.

For any natural number r let $\Phi^{\uparrow r}$ be the formula obtained from the simple formula Φ by shifting all indices by r;
i.e., by replacing each occurrence of every $\mathtt{st}_n$ with $\mathtt{st}_{n+r}$.

We propose the following axioms:

IS (Structural axioms)
For every $S \subseteq S'$, $\mathbb{S}_S \subseteq \mathbb{S}_{S'}$.

For every S, S', S'' of the same cardinality

$$\mathbb{I}_S^{S'} : \mathbb{S}_S \to \mathbb{S}_{S'}, \quad \mathbb{I}_S^{S} = Id_S, \quad \mathbb{I}_{S'}^{S} = (\mathbb{I}_S^{S'})^{-1}, \quad \mathbb{I}_S^{S'} \circ \mathbb{I}_{S'}^{S''} = \mathbb{I}_S^{S''};$$

$$x, z \in \mathbb{S}_S \Rightarrow (x \in z \ \leftrightarrow \ \mathbb{I}_S^{S'}(x) \in \mathbb{I}_S^{S'}(z));$$

$$x \in \mathbb{S}_T \Rightarrow \mathbb{I}_T^{T'}(x) = \mathbb{I}_S^{S'}(x)$$

where $T \subseteq S$ and T' is the image of T by the order-preserving map of S onto S'.

ET (Extended Transfer)
Let ϕ be an $\in$-formula with st_S-parameters. Then

$$\forall^{\mathrm{st}_S} x\ \phi(x) \Rightarrow \forall x\ \phi(x).$$

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HO (Homogeneous Shift)
Let Φ be a simple formula. For any natural number r,

$$\forall^{\mathrm{st}_S} x \; \left(\Phi(x) \; \leftrightarrow \; \Phi^{\uparrow r}(\mathbf{i}_S^{S+r}(x)) \right).$$

$S < T$ stands for $\forall s \in S \, \forall t \in T \, (s < t)$.

EE (End extension)

$$\forall n \in \mathbb{S}_S \cap \mathbb{N} \, (n \in \mathbb{S}_0 \ \vee \ \forall T < S \, \forall m \in \mathbb{S}_T \, (m < n)).$$

SPOTS is the theory **SPOT** $+$ **ET** $+$ **HO** $+$ **IS** $+$ **EE.**
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Proposition

SPOTS interprets Jin's Property 2.1.

Proof. We define:
$\mathbb{V}_0 = \mathbb{S}_0$, $\mathbb{V}_1 = \mathbb{S}_{\{0\}}$, $\mathbb{V}_2 = \mathbb{S}_{\{0,1\}}$, $\mathbb{V}_3 = \mathbb{S}_{\{0,1,2\}}$, and
$i_1 = \mathbb{I}^{\{1\}}_{\{0\}}$, $i_2 = \mathbb{I}^{\{0,2\}}_{\{0,1\}}$, $i_* = \mathbb{I}^{\{1,2\}}_{\{0,1\}}$. □

CONSERVATIVITY

Theorem

SPOT $+$ **SN** is a conservative extension of **ZF**.

Question:
Is **SPOT** $+$ **SC** a conservative extension of **ZF**?

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Theorem

SCOT is a conservative extension of **ZF** $+$ **ADC.**

These results are established by constructions that extend and combine the methods of forcing developed by Ali Enayat and Mitchell Spector.

A. Enayat, *From bounded arithmetic to second order arithmetic via automorphisms*, in: A. Enayat, I. Kalantari, and M. Moniri (Eds.), Logic in Tehran, Lecture Notes in Logic, vol. 26, ASL and AK Peters, 2006
`http://academic2.american.edu/~enayat`

M. Spector, *Extended ultrapowers and the Vopěnka–Hrbáček theorem without choice*, Journal of Symboic Logic 56, 2 (1991), 592–607. `https://doi.org/10.2307/2274701`

The forcing construction used to establish conservativity of **SCOT** over **ZF** $+$ **ADC** is simple.

Definition. Let $\mathbb{P} = \{p \subseteq \mathbb{N} \mid p \text{ is infinite}\}$. For $p, p' \in \mathbb{P}$ we say that p' *extends* p (notation: $p' \leq p$) iff $p' \subseteq p$.

Forcing with $\mathbb{P}$ is equivalent to forcing with $\mathbb{B} = \mathcal{P}^{\infty}(\mathbb{N})/_{\mathsf{fin}}$.

Let $\mathcal{M} = (M, \in^{\mathcal{M}})$ be a countable model of **ZF** $+$ **ADC**.
If $\mathcal{G}$ is a generic filter over $\mathbb{P}^{\mathcal{M}}$, the generic extension $\mathcal{M}[\mathcal{G}]$ is a model of **ZF** $+$ **ADC** and $\mathcal{G}$ is a nonprincipal ultrafilter over $\mathbb{N}^{\mathcal{M}}$.
The forcing does not add any new reals, ie, every set of natural numbers in $\mathcal{M}[\mathcal{G}]$ belongs to M.

Working inside $\mathcal{M}[\mathcal{G}]$ one can construct the ultrapower $(M^{\mathbb{N}}/\mathcal{G}, \in^*)$ of $\mathcal{M}$ by $\mathcal{G}$ in the usual way.

Łoś's Theorem holds in $\mathcal{N}$ because **ACC** is available in $\mathcal{M}$ and $\mathcal{M}$ canonically embeds into $(M^{\mathbb{N}}/\mathcal{G}, \in^*)$.

This construction extends $\mathcal{M}$ to a model $\mathcal{N} = (M^{\mathbb{N}}/\mathcal{G}, \in^*, M)$ of **SCOT**.

The generic filter $\mathcal{G}$ is *M-iterable*. ie, for every $A \in M$, $A \subseteq \mathbb{N}^2$ implies that

$$\{m \in \mathbb{N} \mid \{n \in \mathbb{N} \mid \langle m, n \rangle \in A\} \in \mathcal{G}\} \in M.$$

This enables the definition of
$\mathcal{G} \otimes \mathcal{G} = \{A \subseteq \mathbb{N}^2 \mid \{m \in \mathbb{N} \mid \{n \in \mathbb{N} \mid \langle m, n \rangle \in A\} \in \mathcal{G}\} \in \mathcal{G}$.

The ultrapower $M^{\mathbb{N}^2}/\mathcal{G} \otimes \mathcal{G}$ is isomorphic to $(M^{\mathbb{N}}/\mathcal{G})^{\mathbb{N}}/\mathcal{G}$.

The structure $\big((M^{\mathbb{N}}/\mathcal{G})^{\mathbb{N}}/\mathcal{G}, M^{\mathbb{N}}/\mathcal{G}, M\big)$ has two levels of standardness.

One can iterate the ultrapower any finite number of times. Familiar arguments then show

Theorem

The theory **SCOTS** is a conservative extension of **ZF** + **ADC**.

CONSERVATIVITY OF SPOT OVER ZF

This is significantly more complicated.

Let $\mathbb{Q} = \{q \in \mathbb{V}^{\mathbb{N}} \mid \exists k \in \mathbb{N}\, \forall i \in \mathbb{N}\, (q(i) \subseteq \mathbb{V}^k \wedge q(i) \neq \emptyset)\}$.

The number k is the *rank* of q. We note that $q(i)$ for each $i \in \mathbb{N}$, and q itself, are sets, but $\mathbb{Q}$ is a proper class.

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Let $\mathbb{Q} = \{q \in \mathbb{V}^{\mathbb{N}} \mid \exists k \in \mathbb{N}\, \forall i \in \mathbb{N}\, (q(i) \subseteq \mathbb{V}^k \wedge q(i) \neq \emptyset)\}$.

The number k is the *rank* of q. We note that $q(i)$ for each $i \in \mathbb{N}$, and q itself, are sets, but $\mathbb{Q}$ is a proper class.

The forcing notion $\mathbb{H}$ is defined as follows: $\mathbb{H} = \mathbb{P} \times \mathbb{Q}$ and $\langle p', q' \rangle \in \mathbb{H}$ *extends* $\langle p, q \rangle \in \mathbb{H}$ (notation: $\langle p', q' \rangle \leq \langle p, q \rangle$) iff p' extends p, $\operatorname{rank} q' = k' \geq k = \operatorname{rank} q$, and for almost all $i \in p'$ and all $\langle x_0, \ldots, x_{k'-1} \rangle \in q'(i)$, $\langle x_0, \ldots, x_{k-1} \rangle \in q(i)$.

The poset $\mathbb{P}$ is used to force a generic filter over $\mathbb{N}$ (à la Enayat), and $\mathbb{H}$ forces an extended ultrapower of $\mathbb{V}$ by the generic filter $\mathcal{U}$ forced by $\mathbb{P}$. It is a modification of the forcing notion of Spector, with the difference that $\mathcal{U}$ is not assumed to be a given ultrafilter in $\mathbb{V}$ but it is forced by $\mathbb{P}$.

The *forcing language* $\mathfrak{L}$ has a constant symbol $\dot{G}_n$ for each $n \in \mathbb{N}$.

Given an $\in$-formula $\phi(v_1, \ldots, v_s)$, we define the *forcing relation* $\langle p, q \rangle \Vdash \phi(\dot{G}_{n_1}, \ldots, \dot{G}_{n_s})$ for $\langle p, q \rangle \in \mathbb{H}$ by meta-induction on the logical complexity of ϕ.

We use $\neg, \wedge$ and $\exists$ as primitives and consider the other logical connectives and quantifiers as defined in terms of these.

The notation $\forall^{\mathbf{aa}} i \in p$ (*for almost all* $i \in p$) means $\forall i \in p \setminus c$ for some finite c.

Definition.
(1) $\langle p,q\rangle \Vdash \dot{G}_{n_1} = \dot{G}_{n_2}$ iff $\operatorname{rank} q = k > n_1, n_2$ and $\forall^{\mathbf{aa}} i \in p\, \forall \langle x_0,\ldots,x_{k-1}\rangle \in q(i)\ (x_{n_1} = x_{n_2})$.
(2) $\langle p,q\rangle \Vdash \dot{G}_{n_1} \in \dot{G}_{n_2}$ iff $\operatorname{rank} q = k > n_1, n_2$ and $\forall^{\mathbf{aa}} i \in p\, \forall \langle x_0,\ldots,x_{k-1}\rangle \in q(i)\ (x_{n_1} \in x_{n_2})$.
(3) $\langle p,q\rangle \Vdash \neg\phi(\dot{G}_{n_1},\ldots,\dot{G}_{n_s})$ iff $\operatorname{rank} q = k > n_1,\ldots,n_s$ and, for no $\langle p',q'\rangle \le \langle p,q\rangle$, $\langle p',q'\rangle \Vdash \phi(\dot{G}_{n_1},\ldots,\dot{G}_{n_s})$.
(4) $\langle p,q\rangle \Vdash (\phi \wedge \psi)(\dot{G}_{n_1},\ldots,\dot{G}_{n_s})$ iff $\langle p,q\rangle \Vdash \phi(\dot{G}_{n_1},\ldots,\dot{G}_{n_s})$ and $\langle p,q\rangle \Vdash \psi(\dot{G}_{n_1},\ldots,\dot{G}_{n_s})$.
(5) $\langle p,q\rangle \Vdash \exists v\, \psi(\dot{G}_{n_1},\ldots,\dot{G}_{n_s},v)$ iff $\operatorname{rank} q = k > n_1,\ldots,n_s$ and for every $\langle p',q'\rangle \le \langle p,q\rangle$ there exist $\langle p'',q''\rangle \le \langle p',q'\rangle$ and $m \in \mathbb{N}$ such that $\langle p'',q''\rangle \Vdash \psi(\dot{G}_{n_1},\ldots,\dot{G}_{n_s},\dot{G}_m)$.

Basic properties of forcing:

(1) If $\langle p, q \rangle \Vdash \phi$ and $\langle p', q' \rangle$ extends $\langle p, q \rangle$, then $\langle p', q' \rangle \Vdash \phi$.

(2) No $\langle p, q \rangle$ forces both ϕ and $\neg\phi$.

(3) Every $\langle p, q \rangle$ extends to $\langle p', q' \rangle$ such that $\langle p', q' \rangle \Vdash \phi$ or $\langle p', q' \rangle \Vdash \neg\phi$.

(4) f $\langle p, q \rangle \Vdash \phi$ and $p' \setminus p$ is finite, then $\langle p', q \rangle \Vdash \phi$.

The following proposition establishes a relationship between this forcing and ultrapowers.

"Łoś's Theorem"

Let $\phi(v_1, \ldots, v_s)$ be an $\in$-formula with parameters from $\mathbb{V}$. Then $\langle p, q\rangle \Vdash \phi(\dot{G}_{n_1}, \ldots, \dot{G}_{n_s})$ iff $\operatorname{rank} q = k > n_1, \ldots, n_s$ and $\forall^{\mathbf{aa}} i \in p \, \forall \langle x_0, \ldots, x_{k-1}\rangle \in q(i) \; \phi(x_{n_1}, \ldots, x_{n_s})$.

Let $\mathcal{M} = (M, \in^{\mathcal{M}})$ be a countable model of **ZF**.
Let $\Omega = \{m \in M \mid \mathcal{M} \models \text{“}m \in \mathbb{N}\text{”}\}$.
Let $\mathcal{G}$ be an $\mathcal{M}$-generic filter over $\mathbb{H}$.

We define binary relations $=^*$ and $\in^*$ on Ω as follows:
$m =^* n$ iff there exists $\langle p, q \rangle \in \mathcal{G}$ such that $\operatorname{rank} q = k > m, n$ and $\langle p, q \rangle \Vdash \dot{G}_m = \dot{G}_n$;
$m \in^* n$ iff there exists $\langle p, q \rangle \in \mathcal{G}$ such that $\operatorname{rank} q = k > m, n$ and $\langle p, q \rangle \Vdash \dot{G}_m \in \dot{G}_n$.

It is easily seen from the definition of forcing and “Łoś’s Theorem” that $=^*$ is an equivalence relation on Ω, and a congruence relation with respect to $\in^*$.

Let G_m be the equivalence class of $m \in \Omega$ in the relation $=^*$. Define $G_m \in^* G_n$ iff $m \in^* n$, and let $N = \{G_m \mid m \in \Omega\}$.

The *extended ultrapower of* $\mathcal{M}$ *by* $\mathcal{U}$ is the structure $\mathcal{N} = (N, \in^*)$.

There is a natural embedding j of $\mathcal{M}$ into $\mathcal{N}$: For $x \in M$
$j(x) = G_m$ iff there exists $\langle p, q \rangle \in \mathcal{G}$ such that $\operatorname{rank} q = k > m$ and $\forall^{\mathbf{aa}} i \in p \, \forall \langle x_0, \ldots, x_{k-1} \rangle \in q(i) \; (x_m = x)$.

The Fundamental Theorem of Extended Ultrapowers

Let $\phi(v_1, \ldots, v_s)$ be an $\in$-formula with parameters from M. If $G_{n_1}, \ldots, G_{n_s} \in N$, then the following statements are equivalent:

(1) $\mathcal{N} \models \phi(G_{n_1}, \ldots, G_{n_s})$.

(2) There is some $\langle p, q \rangle \in \mathcal{G}$ such that $\langle p, q \rangle \Vdash \phi(\dot{G}_{n_1}, \ldots, \dot{G}_{n_s})$ holds in $\mathcal{M}$.

(3) There exists some $\langle p, q \rangle \in \mathcal{G}$ with $\operatorname{rank} q = k > n_1, \ldots, n_s$ such that $\mathcal{M} \models \forall i \in p \forall \langle x_0, \ldots, x_{k-1} \rangle \in q(i)\ \phi(x_{n_1}, \ldots, x_{n_s})$.

Corollary. The embedding j is an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$.

Corollary. The structure $\mathcal{N}$ satisfies **ZF**.

Theorem

The structure $\widehat{\mathcal{N}} = (N, \in^*, M)$ satisfies the principles of Transfer, Nontriviality, Standard Part and **SN**.

Corollary. **SPOT** $+$ **SN** is a conservative extension of **ZF**.

Conjecture:
SPOTS is a conservative extension of **ZF**.

The obvious idea is to iterate the forcing used to prove the conservativity of **SPOT**. It is complicated...