Continuous reducibility is a well-quasi-order on the class of continuous functions with Polish 0-dimensional domains

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General question

A **solid** way to compare definable functions on topological spaces:

- topologically relevant,
- combinatorially simple, and
- the finest possible to have these properties.

A general question.

Find solid ways to compare Borel functions on Polish spaces.

- **Polish spaces**, that is separable completely metrizable spaces,
- **Borel** sets are in the smallest σ-algebra generated by open sets,
- The classes $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$ on the α-th level of the Borel hierarchy.
- For any of these pointclasses $\mathbf{\Gamma}$, a function is **$\mathbf{\Gamma}$-measurable** if preimages of open sets are in $\mathbf{\Gamma}$,
- A function is **Borel** if it is Borel-measurable.

General question, more specific

- A **quasi-order** (qo) on a set Q is a reflexive and transitive relation $\leq \subseteq Q^2$.
  - $p < q$ when $p \leq q$ but $q \not\leq p$,
  - ($\leq$-equivalence) $p \equiv q$ when $p \leq q$ and $q \leq p$
- A qo $(Q, \leq)$ is **finer** than $(Q, \leq')$ if $p \leq q$ implies $p \leq' q$.
- Topological relevance: **preserving $\boldsymbol{\Gamma}$-measurability**:
  If $f \leq g$ and g is $\boldsymbol{\Gamma}$-measurable, then so should be f.
- Combinatorial simplicity: **well-quasi-order** (wqo)
  - there should be no infinite strictly descending chain, and
  - no infinite set of pairwise incomparable elements (antichains).

A general question.

Find fine wqos for Borel functions on Polish spaces that preserve $\boldsymbol{\Gamma}$-measurability.

Comparing subsets: Continuous, or Wadge reducibility

Definition (Wadge reduction)

For $A \subseteq X$ and $B \subseteq Y$, we say that A **continuously reduces** to B if there is a continuous function $f : X \to Y$ such that $f^{-1}(B) = A$.

A space is **0-dimensional** if it has a basis of clopen sets.

Theorem (Wadge, Martin-Monk)

Continuous reducibility is a wqo on Borel subsets of Polish 0-dimensional spaces.

What about other spaces?

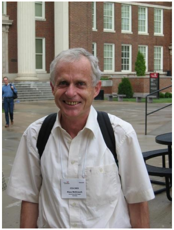
Theorem (Schlicht)

Continuous reducibility has infinite antichains on Borel sets of any Polish non 0-dimensional space.

This is why we choose to focus on **Polish 0-dimensional** spaces.

Weihrauch's continuous reducibility

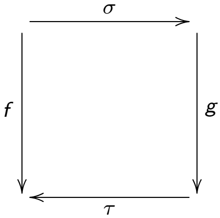
Given topological spaces X, Y, X', Y',
and two functions $f : X \to Y$ and $g : X' \to Y'$,



Definition

A **continuous reduction** from f to g is a pair $(\sigma : X \to X', \tau : \mathrm{Im}(g \circ \sigma) \to Y)$ of continuous functions such that $f = \tau \circ g \circ \sigma$.

Write $f \leq g$ when f reduces continuously to g.

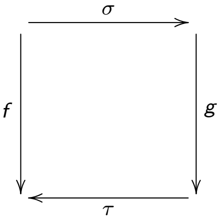


Some first remarks

Continuous reducibility:

Definition

$f \leq g$ iff $f = \tau \circ g \circ \sigma$ for some **continuous maps** σ and τ.



- preserves $\mathbf{\Gamma}$-measurability.
- It is different from reduction on graphs: there are $\mathbf{\Sigma}^0_\alpha$-measurable functions with closed graphs for arbitrarily large $\alpha < \omega_1$.
- $\mathrm{Id}_X \leq \mathrm{Id}_Y$ iff X topologically embeds in Y.
- $\mathbf{1}_{A,X} \leq \mathbf{1}_{B,Y}$ iff $A \leq_W^{X,Y} B$ or $A \leq_W^{X,Y} \neg B$.

The main result

Theorem (Carroy-Pequignot)

Continuous reducibility is a well-quasi-order on continuous functions between Polish 0-dimensional spaces.

Strong Weihrauch reducibility

A similar quasi-order in the context of computable analysis: strong Weihrauch reducibility.
This is for F and G **multi-functions** and (σ, τ) a pair of **computable** functions.

Theorem (Dzhafarov)

Strong Weihrauch reducibility induces a (non-distributive) lattice in which any countable distributive lattice embeds.

So this is (very) far from being wqo...
Continuous reducibility corresponds to topological strong Weihrauch reducibility on single-valued functions.
One says that F is **Weihrauch reducible** to G if there is a pair (σ, τ) such that $F(x) = \tau(x, G \circ \sigma(x))$ for all x.
Topological Weihrauch reducibility makes all continuous functions equivalent! We want finer.

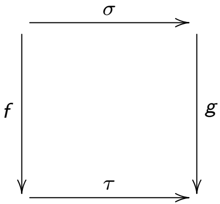
Going finer: Solecki's topological embeddability

Given two functions $f : X \to Y$ and $g : X' \to Y'$

Definition

A **topological embedding** from f to g is a pair $(\sigma : X \to X', \tau : \mathrm{Im}(f) \to Y')$ of continuous embeddings such that $\tau \circ f = g \circ \sigma$.

Note $f \sqsubseteq g$ when f embeds in g.



Some finite basis results for topological embeddability

A subset P of a qo Q has a **finite basis** if there is a finite $B \subseteq P$ such that for all $p \in P$ there is $b \in B$ with $b \leq_Q p$.

Q is wqo $\iff$ all $P \subseteq Q$ have a finite basis.

Classes with a finite basis for topological embeddability:

- Borel functions between Polish spaces that have uncountable image (size 1, $\mathrm{Id}_{\mathbb{N}^\mathbb{N}}$).
- (Solecki, Zapletal, Pawlikovski-Sabok) Borel non σ-continuous functions from an analytic space to a separable metrizable one (size 1! The Pawlikovski function P).
- (Carroy-Miller) All Baire-measurable functions from the Baire space to a separable metric space (size 2).
- (Carroy-Miller) All functions from $\mathbb{Q}$ to a metric space (3).
- (Carroy-Miller) Non Baire class 1 analytic functions from an analytic space to a separable metrizable one (6).

What about maximal functions?

The projection $[0,1]^{\mathbb{N}} \times [0,1]^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ is a maximum for topological embeddability on continuous function between separable metrizable spaces.

Using a generalisation of the Bourgain rank due to Elekes, Kiss, and Vidnyánszky, we see that larger classes don't have maximal elements.

Theorem (Carroy - Pequignot - Vidnyánszky)

For countable $\alpha \neq 0$, there is no $\sqsubseteq$-maximal $\mathbf{\Sigma}^0_{\alpha+1}$-measurable function between Polish 0-dimensional spaces.

The complexity of topological embeddability

On the space $\mathrm{C}(X,Y)$ of continuous functions $X \to Y$ we put the **compact-open topology**, generated by

$$S_{X,Y}(K,U) = \{f \in \mathrm{C}(X,Y) \mid f(K) \subseteq U\},$$

for $K \subseteq X$ compact and $U \subseteq Y$ open.
If X is compact, it is a Polish topology.

Theorem (Carroy - Pequignot - Vidnyánszky)

If X is compact then $(\mathrm{C}(X,Y), \sqsubseteq)$ is a $\mathbf{\Sigma}^1_1$ quasi-order.

Compactness is needed: there is a counter-example using a locally compact domain.

A dichotomy

Theorem (Carroy - Pequignot - Vidnyánszky)

If X has infinitely many limit points, and if Y is not discrete then $(\mathsf{C}(X,Y),\sqsubseteq)$ is a $\mathbf{\Sigma}^1_1$-hard quasi-order.

So, in these cases, topological embeddability reduces every Borel quasi-order, so it is as far from being a wqo as possible.
In the other cases in which X is compact, it turns out to be wqo!

Theorem (Carroy - Pequignot - Vidnyánszky)

If X and Y are Polish 0-dimensional and X is compact then

- *either $(\mathsf{C}(X,Y),\sqsubseteq)$ is a $\mathbf{\Sigma}^1_1$-complete quasi-order,*
- *or it is wqo.*

Continuous reducibility has maximal functions

We call **continuously complete** for a class $\mathcal{C}$ a function that belongs to $\mathcal{C}$ and that reduces all other functions of $\mathcal{C}$.
On Polish 0-dimensional spaces:

- $\mathrm{Id}_{\mathbb{N}^\mathbb{N}}$ is continuously complete for continuous functions
- The limit function lim, mapping a converging sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}^\mathbb{N}$ to its limit, is continuously complete for $\mathbf{\Sigma}^0_2$-measurable functions.
- The Turing Jump is continuously complete for $\mathbf{\Sigma}^0_2$-measurable functions.
- $J^{(\alpha)}$ and $\lim^{(\alpha)}$ are $\mathbf{\Sigma}^0_{\alpha+1}$-measurable continuously complete, for $0 < \alpha < \omega_1$.

Proposition

All $\mathbf{\Sigma}^0_2$-measurable, non σ-continuous functions between Polish spaces are continuously equivalent.

Because $\lim \leq P$... Generalized by Marks-Montalban.

Well-quasi-order results

Theorem (Wadge, Martin-Monk, van Engelen-Miller-Steel)

Continuous reducibility is a well-quasi-order on Borel functions from the Baire space to a finite set.

Theorem (Carroy-Pequignot)

Continuous reducibility is a well-quasi-order on continuous functions between Polish 0-dimensional spaces.

Here is the strategy that we follow. The first step is to get rid of continuous functions with uncountable image.

Proposition

All continuous functions from a Polish 0-dimensional space that have uncountable image are continuously equivalent to $\mathrm{Id}_{\mathbb{N}^\mathbb{N}}$.

So we can focus on continuous functions **with countable image**.

The Cantor-Bendixson rank of a function.

Notation

C denotes the class of continuous functions between Polish 0-dimensional spaces that have countable image.

Say $x \in \mathrm{dom}(f)$ is f-**isolated** if f is locally constant in x.

For $f \in \mathsf{C}$, f-isolated points form a dense open subset of $\mathrm{dom}(f)$, so we can define as usual by induction a decreasing sequence of closed derivatives.

Definition

- $\mathrm{CB}_0(f) = \mathrm{dom}(f)$,
- $\mathrm{CB}_{\alpha+1}(f) = \{x \in \mathrm{CB}_\alpha(f) \mid x \text{ is not } f|_{\mathrm{CB}_\alpha(f)}\text{-isolated}\}$,
- $\mathrm{CB}_\lambda(f) = \bigcap_{\alpha\in\lambda} \mathrm{CB}_\alpha(f)$ for λ limit.

If $f \in \mathsf{C}$, then for some $\alpha < \omega_1$ $\mathrm{CB}_\alpha(f) = \emptyset$.

The Cantor-Bendixson rank of a function.

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Definition

The minimal such α is the **Cantor-Bendixson rank of** f, denoted by $\mathrm{CB}(f)$.

For a closed set F, a point $x \in F$ is isolated iff it is Id_F-isolated, so the usual Cantor-Bendixson rank of F is in fact the rank of Id_F.

The Cantor-Bendixson type of a function.

Notation

For $\alpha < \omega_1$, denote $\mathsf{C}_\alpha = \{f \in \mathsf{C} \mid \mathsf{CB}(f) = \alpha\}$.

If $\mathsf{CB}(f) = \alpha + 1$, then $f|_{\mathsf{CB}_\alpha(f)}$ is locally constant. Denote N_f the cardinal of its image. Define N_f to be 0 if $\mathsf{CB}(f)$ is limit.

Set the **type** of $f \in \mathsf{C}$ to be $tp(f) = (\mathsf{CB}(f), N_f)$.

The type is an invariant for continuous reducibility on C:
for $f, g \in \mathsf{C}$, $f \leq g$ implies $tp(f) \leq_{lex} tp(g)$.
The converse is not true in general, but

Theorem (Carroy)

Suppose that f, g are in C and f has compact domain, then $tp(f) \leq_{lex} tp(g)$ implies $f \leq g$.

A result on the general structure of C.

In particular,

- Continuous reduction is a well-order of length $\omega_1 + 1$ on functions in C that have compact domain,
- If $K \subset \mathbb{N}^{\mathbb{N}}$ is compact of type $(\alpha + 1, 1)$ then Id_K is minimal for functions in C of rank $> \alpha$.

Theorem (Carroy)

Suppose that $f, g \in \mathsf{C}$ satisfy $\mathsf{CB}(f) = \lambda + n$ and $\mathsf{CB}(g) = \alpha$ for some $\lambda < \omega_1$ limit or null, $n \in \omega$, and $\alpha < \omega_1$:

1. *If $n = 0$ and $\mathsf{CB}(f) = \lambda = \alpha = \mathsf{CB}(g)$, then $f \leq g$.*
2. *If $\lambda + 2n < \alpha$, then $f \leq g$.*

As a consequence,

Corollary

If C_α is wqo for all $\alpha < \omega_1$ then C is wqo.

The finite generation method.

Suppose that $A_i \subset \mathbb{N}^{\mathbb{N}}$ for all $i \in \mathbb{N}$, then the **gluing of the sets** A_i is $\oplus_i A_i = \bigcup_i (i)^\frown A_i$.

Similarly if $f_i : A_i \to B_i$ for all $i \in \mathbb{N}$ then the **gluing of the functions** f_i is $\oplus_i f_i : \oplus_i A_i \to \oplus_i B_i$, $(i)^\frown x \mapsto (i)^\frown f_i(x)$.

A class $\mathcal{C}$ of functions is **finitely generated** if there is a finite set G such that every function of $\mathcal{C}$ is (continuously equivalent to) a finite gluing of functions in G.

Theorem (Carroy-Pequignot)

The class C_α is finitely generated for all $\alpha < \omega_1$.

Since any finitely generated class is a well-quasi-order under continuous reducibility, continuous functions are well-quasi-ordered under continuous reducibility.

A summarizing table.

Here C abbreviates $\mathrm{C}(X, Y)$ and $\mathbf{\Sigma}^0_2(X, Y)$ stands for the $\mathbf{\Sigma}^0_2$-measurable functions.
LC means locally compact and cpt means compact, while $\mathbf{\Sigma}^1_1$-c. and $\mathbf{\Sigma}^1_1$-h. mean analytic complete and analytic hard, respectively.

<table>
<tr><th rowspan="3"></th><th colspan="2">$\omega^2 \not\sqsubseteq X$</th><th colspan="4">$\omega^2 \sqsubseteq X$</th></tr>
<tr><th rowspan="2">X is LC</th><th rowspan="2">X not LC</th><th rowspan="2">$|Y| < \aleph_0$</th><th rowspan="2">$Y \cong \omega$</th><th colspan="2">$\omega + 1 \sqsubseteq Y$</th></tr>
<tr><th>X cpt</th><th>X not cpt</th></tr>
<tr><td>$(\mathrm{C}, \sqsubseteq)$</td><td>WQO</td><td>WQO?</td><td colspan="2">WQO</td><td>$\mathbf{\Sigma}^1_1$-c.</td><td>$\mathbf{\Sigma}^1_1$-h.</td></tr>
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<tr><td>$(\mathrm{C}, \leq)$</td><td colspan="6">WQO</td></tr>
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</table>

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	X is LC	X not LC	$\lvert Y\rvert < \aleph_0$	$Y \cong \omega$	$\omega + 1 \sqsubseteq Y$	
					X cpt	X not cpt
$(\mathsf{C}, \sqsubseteq)$	WQO	WQO?	WQO		$\mathbf{\Sigma}^1_1$-c.	$\mathbf{\Sigma}^1_1$-h.
$(\mathbf{\Sigma}^0_2, \sqsubseteq)$	?		WQO	$\mathbf{\Sigma}^1_1$-h?	$\mathbf{\Sigma}^1_1$-h.	
$(\mathsf{C}, \leq)$	WQO					
$(\mathbf{\Sigma}^0_2, \leq)$	?		WQO	?		

Thank you!