Mirna Džamonja, IRIF, Paris

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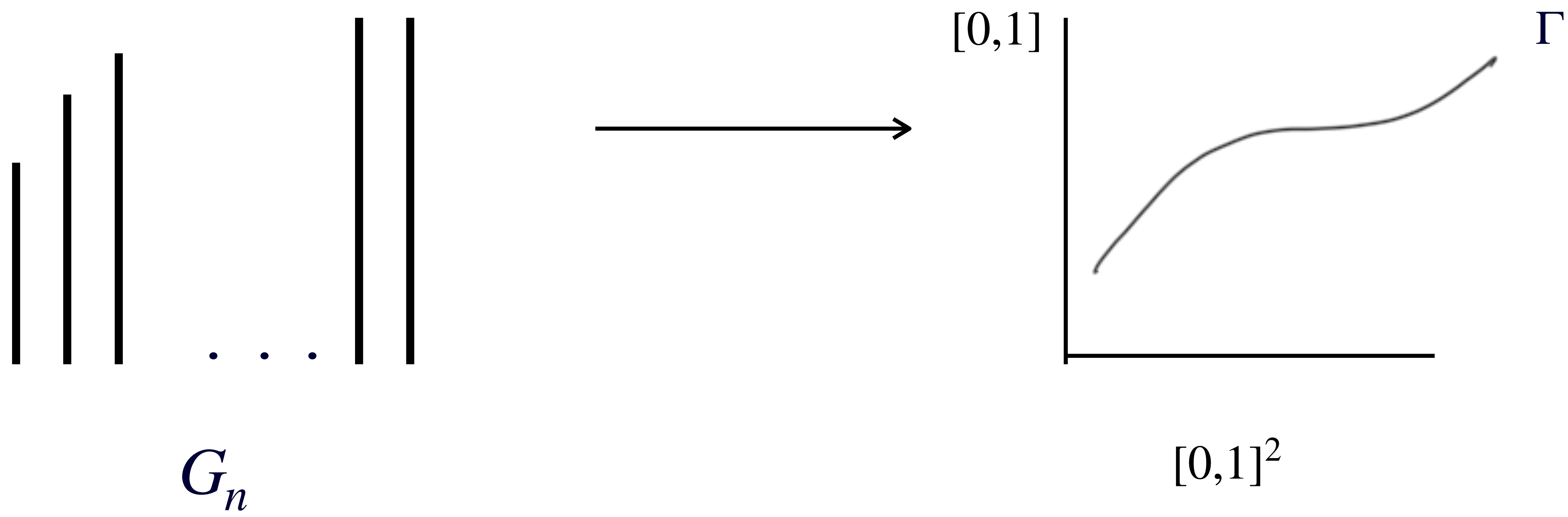
1. Introduction


[0,1]
Γ
G_n
$[0,1]^2$

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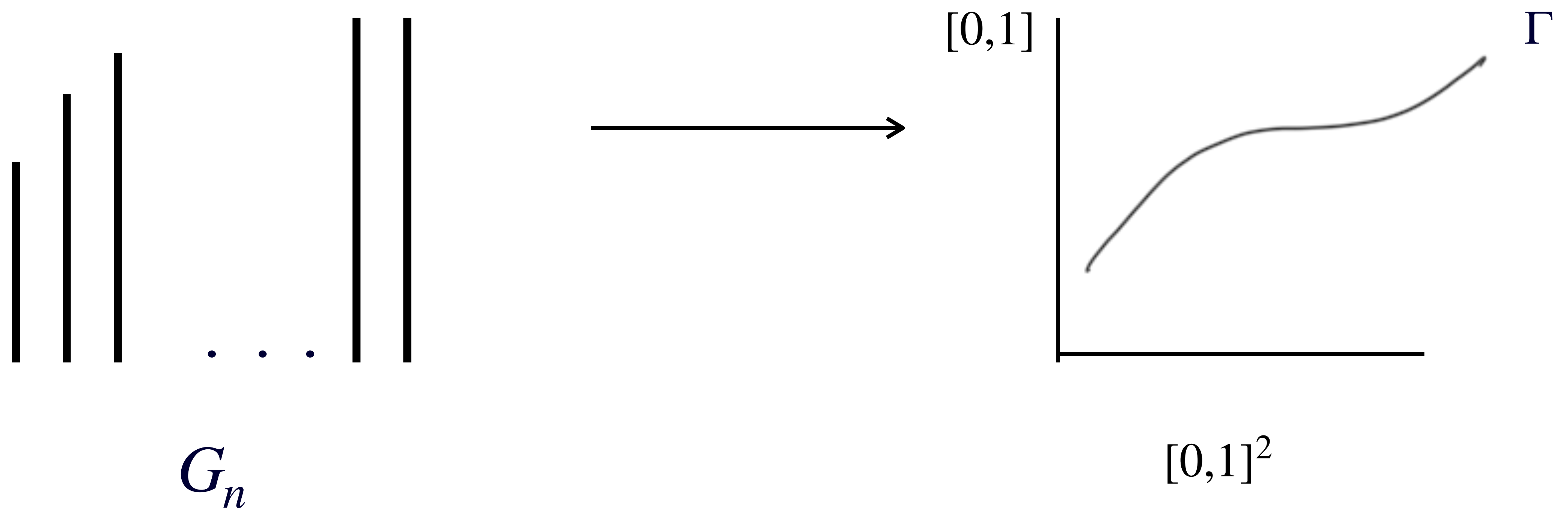

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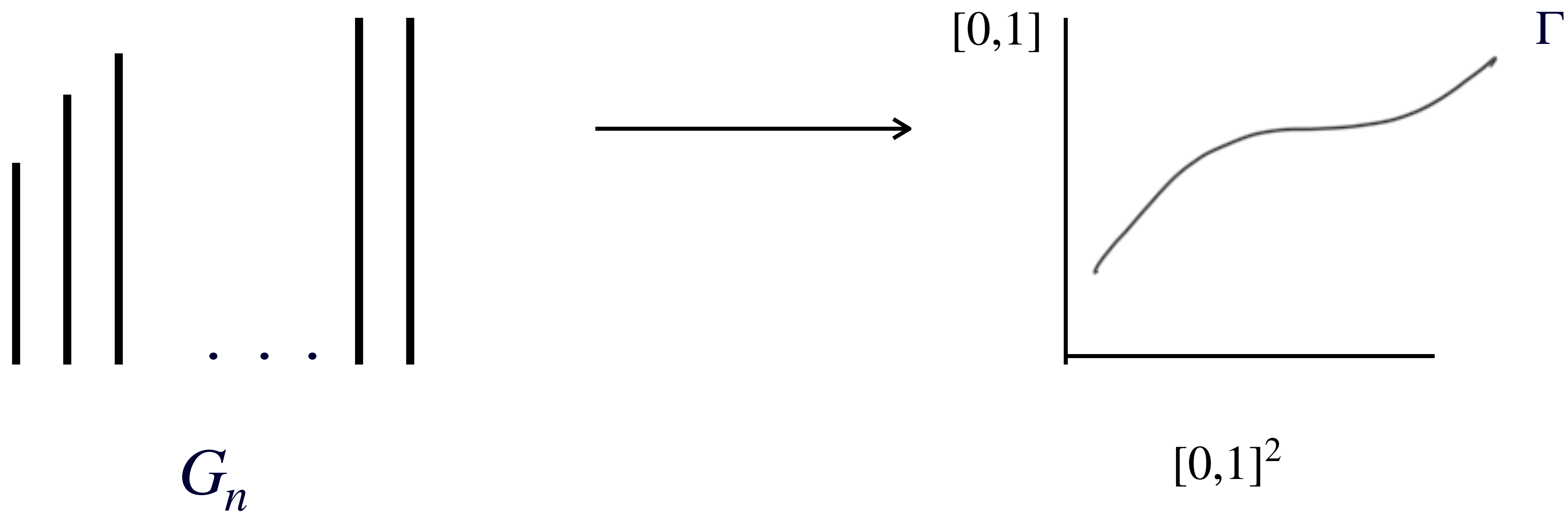

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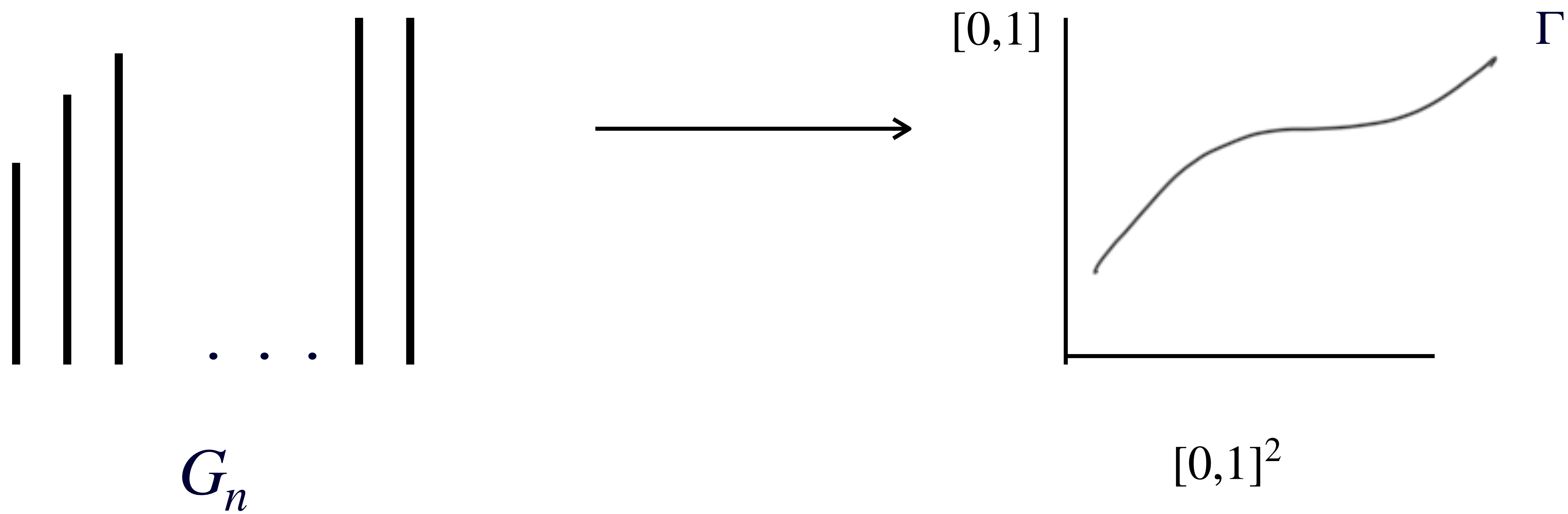

In fact, a graphon is a a measurable function $\Gamma : [0,1] \times [0,1] \to [0,1]$ *which* represents the sequence $\langle G_n : n < \omega \rangle$ *in the sense that certain graph invariants are transferred.*

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There is a notion of metric convergence for the sequence $\langle G_n : n < \omega \rangle$ *associated to this,* cut metric.

COLLOQUIUM PUBLICATIONS

American Mathematical Society

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Volume 60

Large Networks and Graph Limits

László Lovász

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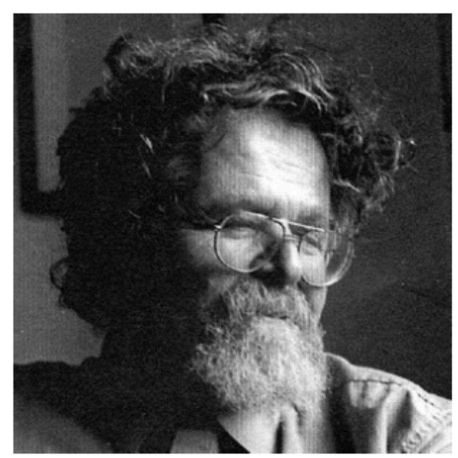
Project: Dynamics and Structure of Networks (DYNASNET)
ERC funding: 9.315 million for 6 years
Researchers and Host institutions:

Albert-László Barabási
Central European University, Budapest

Laszlo Lovasz
Hungarian Academy of Science



Jaroslav Nesetril
Charles University in Prague

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works well for sequences of dense graphs but is rather information-free for sparse graphs, as we get 0 *in the limit.*

To capture sparse graphs, a new theory was needed, developed by Benjamini-Schramm and further by Nešetřil and Osona de Mendez. A unifying theory was given by the latter authors through the notion of

FIRST ORDER CONVERGENCE

which leads to the limit notion called modeling. *In the case of a sequence of dense graphs, a modeling reduces to a graphon.*

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Number 1272

A Unified Approach to Structural Limits and Limits of Graphs with Bounded Tree-Depth

Jaroslav Nešetřil
Patrice Ossona de Mendez

January 2020 • Volume 263 • Number 1272 (second of 7 numbers)

AMERICAN MATHEMATICAL SOCIETY

2. Connections to model theory

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In various situations there is a standard Borel space (so uncountable) A which is a τ-structure and which satisfies $\langle \varphi, A \rangle = \lim_{n \to \infty} \langle \varphi_n, A \rangle$ for all φ. This is the modeling. The notion encapsulates graphons.

2.2. Ultrapowers and Loeb measures

Ultraproducts → Graphons

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While developing the notion of a hypergraphon, Elek and Szegedy (2010) considered an ultraproduct $\Pi_{n\in\omega}(H_n, \mu_n)/\mathcal{U}$, where μ_n is the counting measure on H_n, $\mathcal{U}$ is a non-principal ultrafilter on ω and the hypergraphon is obtained through a certain separable quotient.

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This idea has been extended to measure preserving actions by Conley, Kechris and Tucker-Drob in *Ultraproducts of measure preserving actions and graph combinatorics (2012)*

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Model theory of such objects is well understood (see the work of Hrushovski and others)and has been used to obtain deep combinatorial results (for example the work of Chernikov). We shall review one, and then mention a result of ours with Tomašić that connected that with graphons.

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Tao (2012) proved Tao's algebraic regularity Lemma,

as shown on the next slide.

Lemma 1 (Algebraic regularity lemma) Let F be a finite field, let V, W be definable non-empty sets of complexity at most M, and let $E \subset V \times W$ also be definable with complexity at most M. Assume that the characteristic of F is sufficiently large depending on M. Then we may partition $V = V_1 \cup \ldots \cup V_m$ and $W = W_1 \cup \ldots \cup W_n$ with $m, n = O_M(1)$, with the following properties:

- (Definability) Each of the $V_1, \ldots, V_m, W_1, \ldots, W_n$ are definable of complexity $O_M(1)$.
- (Size) We have $|V_i| \gg_M |V|$ and $|W_j| \gg_M |W|$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$.
- (Regularity) We have

$$|E \cap (A \times B)| = d_{ij}|A||B| + O_M(|F|^{-1/4}|V||W|) \qquad (2)$$

for all $i = 1, \ldots, m$, $j = 1, \ldots, n$, $A \subset V_i$, and $B \subset W_j$, where d_{ij} is a rational number in $[0, 1]$ with numerator and denominator $O_M(1)$.

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(suggested in the private correspondance of Hrushovski to Tao)

What is an asymptotic class ?

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Definition 3.5. Let $\mathcal{C}$ be a class of finite structures (considered a category with substructure embeddings). We say that $\mathcal{C}$ is an *asymptotic class* (in the sense of [8] and [3]), if, for every definable set $\mathbf{X}$ over $\mathbf{S}$, there exist

(1) a definable function $\boldsymbol{\mu}_{\mathbf{X}} : \mathbf{S} \to \mathbb{Q}$,
(2) a definable function $\mathbf{d}_{\mathbf{X}} : \mathbf{S} \to \mathbb{N}$,

so that, for every $\epsilon > 0$ there exists a constant $N > 0$ such that for every $F \in \mathcal{C}$ with $|F| > N$ and every $s \in \mathbf{S}(F)$,

$$\big||\mathbf{X}_s(F)| - \boldsymbol{\mu}_{\mathbf{X}}(s)|F|^{\mathbf{d}_{\mathbf{X}}(s)}\big| \leq \epsilon |F|^{\mathbf{d}_{\mathbf{X}}(s)}.$$

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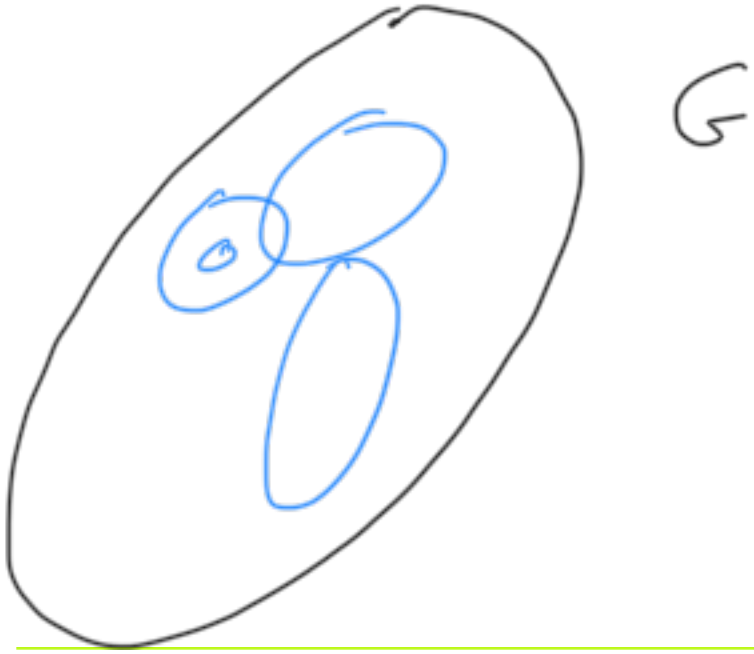

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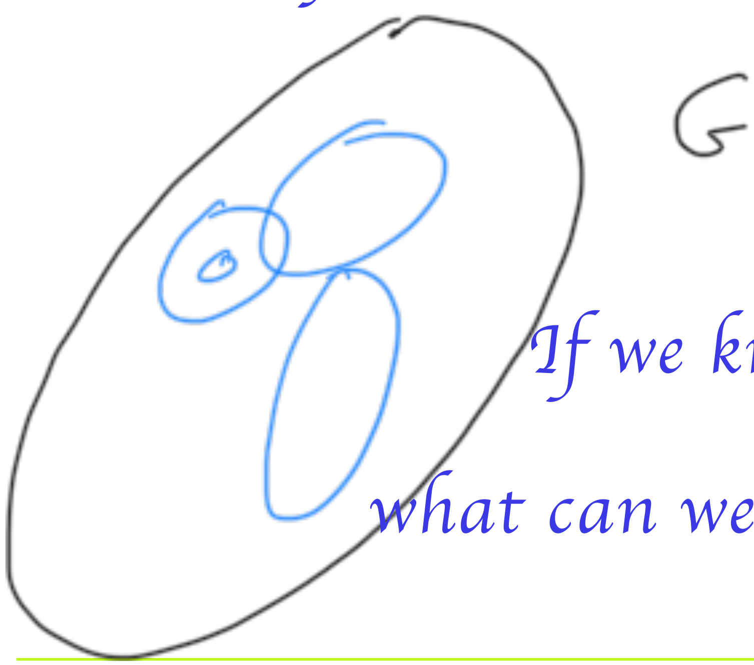

If we know a model-theoretic classification of G,

what can we say about the graphons generated by Age(G)?

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Fact. *Stable graphs are NIP.*

Paper with D. Bartosova, L. Scow and R. Patel, to appear

Theorem 4.5. *Let $\mathcal{L}$ be a finite relational language, $\mathscr{D}$ a nonprincipal ultrafilter over ω, and $(\mathcal{M}_t)_{t\in\omega}$ a $\mathscr{D}$-trending sequence of finite $\mathcal{L}$-structures. Define $\mathfrak{M}^* := \prod_{t\in\omega} \mathcal{M}_t/\mathscr{D}$. Let $\mathfrak{N}$ be an $\mathcal{L}$-structure of cardinality at most $\aleph_1$ such that*

$$\mathcal{K} := \mathrm{age}(\mathfrak{N}) \subseteq \bigcup\{\mathrm{age}(\mathcal{M}_t) : t \in \omega\}.$$

Fix $\mathcal{A} \in \mathcal{K}$ and suppose $\mathcal{A}$ has finite small Ramsey degree in $\mathcal{K}$, with $d := d(\mathcal{A}, \mathcal{K})$. Then for any $k \in \omega$, the partition relation $\mathfrak{M}^* \longrightarrow_{\mathrm{int}} (\mathfrak{N})^{\mathcal{A}}_{k,d}$ holds.

3. Countable versus uncountable limits

The discussion about the connection between the properties of a hereditary class versus the shape of the of graphon space that it generates, illustrates that there is a connection between the countable limits and the uncountable ones.

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The connection exists but is not simple.

Idea: change the countable limit to better reflect the properties of the uncountable limit, notably through changing the logic.

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Most people would answer that it is some syntactical way of generating formulas, usually through a recursive definition, such as FO logic, connected to a semantical notion of interpreting these formulas, again recursively, by a definition like Tarski (Vaught)'s definition of truth.

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However, in abstract model theory, a subject indeed started by Tarski and Vaught in the 1950s, there is much more variety as to what a logic might be and the semantic and syntax are not necessarily connected. We were much inspired by the work of Karol Carp from 1959 to 1974, on chain logic. (Chain logic has nothing to do with this context, it was invented for singular cardinals).

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In a Džamonja-Väänänen paper on connections between chain logics and Shelah's logic L^1_κ (accepted mod. revisions to Israel Journal of Mathematics), we used the following way of framing abstract logics and a way to compare them using Chu transforms. The concepts in the abstract were studied by Garica-Matos and Väänänen (2005).

Definition 1.1 *A* logic *is a triple of the form* $\mathfrak{L} = (L, \models_{\mathfrak{L}}, S)$ *where* $\models_{\mathfrak{L}} \subseteq S \times L$ *and* S *comes with a notion of isomorphism, usually understood from the context. We think of* L *as the set or class of sentences of* $\mathfrak{L}$, S *as a set or class of models of* $\mathfrak{L}$ *and of* $\models_{\mathfrak{L}}$ *as the satisfaction relation. The classes* L *and* S *can be proper classes.*

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Following Shelah:

Definition 1.2 *A logic* $(L, \models_{\mathfrak{L}}, S)$ *is* nice *iff it satisfies the following requirements:*

- *for any* n*-ary relation symbol* P *and constant symbols* $c_0, \ldots c_{n-1}$ *in* τ, $P[c_0, \ldots c_{n-1}]$ *is a sentence in* L,
- L *is closed under negation, conjunction and disjunction,*
- *for any* $\varphi \in L$ *and* $M \in S$, $M \not\models_{\mathfrak{L}} \varphi$ *if and only if* $M \models_{\mathfrak{L}} \neg\varphi$,

Definition 1.1 *A* logic *is a triple of the form* $\mathfrak{L} = (L, \models_{\mathfrak{L}}, S)$ *where* $\models_{\mathfrak{L}} \subseteq S \times L$ *and* S *comes with a notion of isomorphism, usually understood from the context. We think of* L *as the set or class of sentences of* $\mathfrak{L}$, S *as a set or class of models of* $\mathfrak{L}$ *and of* $\models_{\mathfrak{L}}$ *as the satisfaction relation. The classes* L *and* S *can be proper classes.*

Following Shelah:

Definition 1.2 *A logic* $(L, \models_{\mathfrak{L}}, S)$ *is* nice *iff it satisfies the following requirements:*

- *for any* n*-ary relation symbol* P *and constant symbols* $c_0, \ldots c_{n-1}$ *in* τ, $P[c_0, \ldots c_{n-1}]$ *is a sentence in* L,
- L *is closed under negation, conjunction and disjunction,*
- *for any* $\varphi \in L$ *and* $M \in S$, $M \not\models_{\mathfrak{L}} \varphi$ *if and only if* $M \models_{\mathfrak{L}} \neg\varphi$,
- $M \models_{\mathfrak{L}} \varphi_1 \wedge \varphi_2$ *iff* $M \models_{\mathfrak{L}} \varphi_1$ *and* $M \models_{\mathfrak{L}} \varphi_2$, *and similarly for disjunction,*
- *for any* $M \in S$, $a \in M$ *and a sentence* $\psi[a] \in S$ *such that* $M \models_{\mathfrak{L}} \psi[a]$, *we have that* $M \models_{\mathfrak{L}} (\exists x)\psi(x)$, *and conversely, if* $M \models_{\mathfrak{L}} (\exists x)\psi(x)$ *then there is* $a \in M$ *such that* $M \models_{\mathfrak{L}} \psi[a]$,
- *if* M_0 *and* M_1 *are isomorphic models of* τ, *by some isomorphism* f, *and if both* M_0, M_1 *are in* S, *then for every* $\varphi \in L$ *we have* $M_0 \models_{\mathfrak{L}} \varphi[a_0, \ldots a_{n_1}]$ *iff* $M_1 \models_{\mathfrak{L}} \varphi[f(a_0), \ldots f(a_{n_1})]$.

I have been interested to use these ideas to introduce new logics on countable models which will be used to relate them to uncountable models obtained as combinatorial limits. The following is my work in progress on this subject.

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Therefore we obtain a way to interpret the ultrafilter through a countable model.

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Lemma : If there is a modeling A of $\langle M_n : n < \omega \rangle$, then

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So now we have a countable 'mirror' of the uncountable modeling.

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Definition. Let $\mathscr{L} = (\mathscr{L}, \models, \mathscr{S})$ and $\mathscr{L}' = (\mathscr{L}', \models', \mathscr{S}')$ be two logics. We say that $(L, \models, S) \leq (L', \models', S')$ iff there is a pair of functions (f, g) such that $f: L \to L'$, $g: S' \to S$ onto, and the adjointness condition holds, which means $M' \models' f(\varphi) \iff g(M') \models \varphi$

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Heuristic truth supported by various theorems. If $\mathscr{L} \leq \mathscr{L}'$, then the "nice properties" of $\mathscr{L}'$ are inherited by $\mathscr{L}$.

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Transfer principles à la Łos ...