A limiting result for the Ramsey theory of functional equations

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Schur's Theorem

In 1916 Issai Schur published a paper in which he proved:

Theorem (Schur - 1916)

Given any $r \in \mathbb{N}$ *there is a* $S(r) \in \mathbb{N}$ *such that for all* $c : \{1, \ldots, S(r)\} \to \{1, \ldots, r\}$ *one can find* $a, b \in \{1, \ldots, S(r)\}$ *satisfying*

$$c(a) = c(b) = c(a + b).$$

Nomenclature

For each set S and $r \in \mathbb{N}$, any function $C : S \to \{1, \ldots, r\}$ is called r-**colouring of** S; a colouring of S is an r-colouring for some $r \in \mathbb{N}$.

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Theorem (Schur - 1916)

Given any $r \in \mathbb{N}$ there is a $S(r) \in \mathbb{N}$ such that for all r-colouring c of $[S(r)]$ one can find c-monochromatic $a, b, c \in [S(r)]$ such that $a + b = c$.

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Theorem

For all $r \in \mathbb{N}$ *and all* r*-colouring* c *of* $\mathbb{N}$ *one can find* c*-monochromatic* $x, y, z \in \mathbb{N}$ *such that* $x + y = z$*.*

van der Waerden's Theorem

Theorem (van der Waerden – 1927)

Fixed $r \in \mathbb{N}$ and a r-colouring c of $\mathbb{N}$. Then for all $k \in \mathbb{N}$ one can find $a, b \in \mathbb{N}$ such that $a, a+b, a+2b, \cdots, a+(k-1)b$ are c-monochromatic.

Brauer's Theorem

Theorem (A. Brauer – 1928)

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Clearly, we have that Brauer's Theorem implies both van der Waerden's Theorem and Schur's Theorem.

Rado's Theorem

Let A be a $m \times n$ matrix with rational entries. Let $A_1, \ldots, A_m \in \mathbb{Z}^n$ be the columns of A. We say that A satisfies the **columns condition** if there is a partition $I_0, \ldots, I_r$ of $[m]$ such that

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Theorem (R. Rado - 1933)

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1. *Given any colouring c of $\mathbb{N}$, there are c-monochromatic $a_1, \dots, a_n \in \mathbb{N}$ such that $A(a_1, \dots, a_n)^T = \mathbf{0}$; and*

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2. *A satisfies the columns condition.*

Corollary

Let $c_1, \ldots, c_n \in \mathbb{Z}^\times$. The following are equivalent:

1. *Given any colouring c of $\mathbb{N}$, there are c-monochromatic $a_1, \ldots, a_n$ such that $c_1 a_1 + \cdots + c_n a_n = 0$; and*
2. *there is a non-empty $I \subseteq [n]$ such that $\sum_{i \in I} c_i = 0$.*

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   - *there is an $a \in \mathbb{N}$ such that $A(a, a, \ldots, a)^T = \boldsymbol{b}$; or*
   - *there is an $a \in \mathbb{Z}$ such that $A(a, \ldots, a)^T = \boldsymbol{b}$ and A satisfies the columns condition.*

Partition Regularity of Equations

Given any $n \in \mathbb{N}$, any infinite set R, functions $f_1, \ldots, f_m : R^n \to R$, and $r_1, \ldots, r_m \in R$

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$$\begin{cases} f_1(x_1, \ldots, x_n) & = & r_1 \\ \vdots & \vdots & \vdots \\ f_1(x_1, \ldots, x_n) & = & r_m \end{cases}$$

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is **partition regular** (abbr. as **PR**) over an infinite subset $S \subseteq R$ if and only if for all colouring c of R one can find c-monochromatic $s_1, \dots, s_m \in S$ satisfying $f_j(s_1, \dots, s_n) = r_j$ for all $j \in [m]$.

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Theorem (B. van der Waerden – 1922)

The system for any integer $k \geq 3$ *the system there is a* $b \in \mathbb{N}$ *such that the system*

$$\begin{cases} x_2 - x_1 & = & b \\ x_3 - x_2 & = & x_2 - x_1 \\ \vdots & \vdots & \vdots \\ x_k - x_{k-1} & = & x_{k-1} - x_{k-2} \end{cases}$$

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- *there is an $a \in \mathbb{N}$ such that $A(a, a, \ldots, a)^T = \boldsymbol{b}$; or*
- *there is an $a \in \mathbb{Z}$ such that $A(a, \ldots, a)^T = \boldsymbol{b}$ and A satisfies the columns condition.*

Question

What properties on $P_1, \ldots, P_m \in \mathbb{Z}[x_1, \ldots, x_n]$ *could ensure that the system (called system of* ***Diophantine equations****)*

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is PR over $\mathbb{N}$?

The case of linear systems (both homogeneous and inhomogeneous) was completely solved by Rado. But only scarce nonlinear cases are known to be (or not) PR over $\mathbb{N}$.

Hilbert's 10th Problem

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There is no general algorithm that, for any given Diophantine equation, can decide whether this equation has a solution where all the unknowns take integer values.

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Question

Is there an general algorithm that, given a Diophantine equation, can decide whether this equation is PR over $\mathbb{N}$*?*

Open questions

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Given any 2-colouring of $\mathbb{N}$ *one can find c-monochromatic* x, y, z, w *such that* $x + y = z$ *and* $xy = w$*.*

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Theorem (M. Bowen and M. Sabok – 2022)

The system

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is PR over $\mathbb{Q}$

Question (Monochromatic Pythagorean Triples)

Is the equation $x^2 + y^2 = z^2$ partition regular over $\mathbb{N}$?

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Theorem (M. Heule, O. Kullmann and V. W. Marek – 2016)

For every 2-colorings of $\mathbb{N}$ there are c-monochromatic $a, b, c \in \mathbb{N}$ such that $a^2 + b^2 = b^2$.

Some positive results

- (Multiplicative Rado) Given $c_1, \dots, c_m \in \mathbb{Z}^\times$, the equation

$$\prod_{i=1}^{m} x_i = 1$$

is PR over $\mathbb{N}$ iff there is a non-empty $I \subseteq [m]$ such that $\sum_{i \in I} c_i = 0$;

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- (Lefmann) Given any $k \in \mathbb{N}$,

$$c_1 x_1^{\frac{1}{k}} + \cdots + c_m x_m^{\frac{1}{k}} = 0$$

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- (V. Bergelson, H. Furstenberg, McCutcheon) given any $P \in \mathbb{Z}[x]$ such that $P(0) = 0$, the equation

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Some positive results

- (L. Luperi Baglini) Let $m, n \in \mathbb{N}$ $c_1, \ldots, c_n \in \mathbb{Z} \setminus \{0\}$ and $F \subseteq \{1, \ldots, m\}$. If there is a non-empty $I \subseteq \{1, \ldots, m\}$ such that $\sum_{i \in I} c_i = 0$, then the equation

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- (J. Moreira – 2018) Let $c_1, \ldots, c_n \in \mathbb{Z} \setminus \{0\}$ such that $c_1 + \cdots + c_n = 0$. Then the equation $c_1 x_1^2 + \cdots + c_n x_n^2 = y$ is PR over $\mathbb{N}$;

Some positive results

- (M. Di Nasso and L. Luperi Baglini – 2018)
  - If $P \in \mathbb{Z}[y_1, \ldots, y_n]$ has no constant term and $c_1, \ldots, c_m \in \mathbb{Z}^\times$ are such that one can find an $I \subseteq [m]$ satisfying $\sum_{i \in I} c_i = 0$, then the equation

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    $$c_1 x_1 + \cdots + c_m x_m = P(y)$$

    is PR over $\mathbb{N}$ iff there is a non-empty $I \subseteq [m]$ such that $\sum_{i \in I} c_i = 0$;

Some positive results

- (S. Chow, S. Lindqvist, S. Prendiville) For every $n \in \mathbb{N}$ there is a $s(n) \geq 3$ such that

$$c_1 x_1^n + \cdots + c_{s(n)} x_{s(n)}^n = 0$$

is PR over $\mathbb{N}$ if and only if there is a non-empty $I \subseteq [m]$ satisfying $\sum_{i \in I} c_i = 0$. Moreover, $s(2) = 5$ and $s(3) = 8$.

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- (S. Farhangi and R. Magner – 2022) if $a + b \neq 0$ then $ax + by = cw^m z^n$ is not PR over $\mathbb{Z}^\times$

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An ultrafilter on a set S is any collection $\mathcal{U}$ of subsets of S satisfying

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Property (4) above is equivalent to if $A \cup B \in \mathcal{U}$, then either $A \in \mathcal{U}$ or $B \in \mathcal{U}$.

Given any $s \in S$,

$$\mathcal{U}_s = \{A \subseteq S : s \in A\}$$

is an ultrafilter on S, called **principal**. Any non-principal ultrafilter is called **free**.

The set of all ultrafilters of S is βS. For each $A \subseteq S$, let

$$\overline{A} = \{\mathcal{U} \in \beta S : A \in \mathcal{U}\}.$$

Then $\{\overline{A} : A \subseteq S\}$ is a base of open-and-closed subsets for a topology of βS.

The Čech-Stone compactification

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Then $\{\overline{A} : A \subseteq S\}$ is a base of open-and-closed subsets for a topology of βS. Furnished with this topology, βS is completely characterised as the compact (= Hausdorff+compact) space that contains S as a dense subset (identifying every $s \in S$ with $\mathcal{U}_s$) and given any compact K and any $f : S \to K$ there is an unique continuous $\overline{f} : \beta S \to K$ such that $\overline{f}|_S = f$

Ultrafilters on a semigroup

If S has an associative operation $*$, then one can extend this operation to βS as follows:

$$A \in \mathcal{U} * \mathcal{V} \iff \{s \in S : s^{-1}A \in \mathcal{V}\} \in \mathcal{U},$$

where $s^{-1}A = \{t \in S : st \in A\}$.

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where $s^{-1}A = \{t \in S : st \in A\}$. This operation is associative and for all $s \in S$ and $\mathcal{U} \in \beta S$, the maps

$$\beta S \ni \mathcal{V} \mapsto s * \mathcal{V} \quad \text{and} \quad \beta S \ni \mathcal{V} \mapsto \mathcal{V} * \mathcal{U}$$

are continuous

Theorem (Ellis)

Under the above conditions, $\beta S \setminus S$ contains an idempotent element with respect to $*$; i.e. there is an $\mathcal{U} \in \beta S \setminus S$ such that $\mathcal{U} * \mathcal{U} = \mathcal{U}$.

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Let $\mathcal{U} \in \beta S \setminus S$ be idempotent. Then $A \in \mathcal{U} \iff A \in \mathcal{U} * \mathcal{U}$. As such

$$B = A \cap \{s \in S : s^{-1}A \in \mathcal{U}\} \in \mathcal{U}.$$

Thus $B \neq \emptyset$ and given any $s \in B$, $s^{-1}A \cap A \neq \emptyset$. Picking $t \in s^{-1}A \cap A$ we have that $st \in A$.

Galvin-Glazer argument

Hence any idempotent $\mathcal{U}$ in βS satisfies the following: any $A \in \beta S$ contains elements s and t such that $st \in A$.

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Hence, given a colouring c of S, there are $s, t \in S$ such that s, t, st are c-monochromatic.

PR via ultrafilters

Theorem

Given any $n \in \mathbb{N}$, any infinite set R, functions $f_1, \ldots, f_m : R^n \to R$, and $r_1, \ldots, r_m \in R$ system of functional equations

$$\begin{cases} f_1(x_1, \ldots, x_n) & = & r_1 \\ \vdots & \vdots & \vdots \\ f_1(x_1, \ldots, x_n) & = & r_m \end{cases}$$

*is PR over an infinite set $S \subseteq R$ if and only if there is an $\mathcal{U} \in \beta S$ such that for all $A \in \mathcal{U}$ one can find $a_1, \ldots, a_n \in A$ satisfying $f_j(a_1, \ldots, a_n) = s_j$ for all $j \in [m]$. We call such an ultrafilter a **witness** of the PR of the system.*

Non-constant solutions

A **constant solution** for

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Actually, every equation whose PR is known shown so far is non-trivially PR

Main Result

Theorem (Main Result)

Let S be an infinite set, let $n \in \mathbb{N}$ and let $f_1, \ldots, f_m : S^{n+1} \to S$. Let $\sigma(x_1, \ldots, x_n, x_{n+1}) = 0$ be the system of functional equations

$$\begin{cases} f_1(x_1, \ldots, x_n, x_{n+1}) & = & s_1; \\ & \vdots & \\ f_m(x_1, \ldots, x_n, x_{n+1}) & = & s_m. \end{cases}$$

Suppose that there exists $k \in \mathbb{N}$ such that for all $s \in S$ the number of solutions in the variables $x_1, \ldots, x_n$ of $\sigma(x_1, \ldots, x_n, s) = 0$ is at most k. Then the system $\sigma(x_1, \ldots, x_n, x_{n+1}) = 0$ is partition regular on S if and only if it is trivially PR.

Theorem

Let $P_1, \ldots, P_m \in \mathbb{Z}[x, y]$ *be polynomials having degree* ≥ 1 *and*

$$\sigma(x, y) = \begin{cases} P_1(x, y), \\ \vdots \\ P_m(x, y). \end{cases}$$

The following facts are equivalent:

1. *The system* $\sigma(x, y) = 0$ *has a constant solution;*
2. *The system* $\sigma(x, y) = 0$ *is PR on* $\mathbb{N}$.

Proof: In each integral domain R an univariate polynomial P has at most $\deg P$ roots.

PR of equations in two variables

Corollary

In the same notations and hypotheses of the previous Theorem, the following are equivalent:

1. *The system* $\sigma(x, y) = 0$ *is infinitely PR over* $\mathbb{N}$;
2. $(x - y)$ *divides* $P_1(x, y), \ldots, P_m(x, y)$.

In particular, $x - y$ *is the only irreducible infinitely PR polynomial in two variables.*

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Proof: Bézout's Theorem.

PR of S-unit equation

Theorem

Let Γ be a non-torsion multiplicative subgroup of $\mathbb{C}^\times$ of rank r. For all $a, b \in \mathbb{C}^\times$ the equation $ax + by = 1$ has at most $2^{16(r+1)}$ solutions in Γ.

Corollary

Let Γ be a non-torsion multiplicative subgroup of $\mathbb{C}^\times$ of rank r. Given any $a, b, c \in \mathbb{C}^\times$, the equation $ax + by + cz = 0$ is PR over Γ if and only if it has constant solutions, namely if and only if $a + b + c = 0$.

Proof: For all $s \in \Gamma$ we have that
$ax + by + cs = 0 \iff \frac{a}{-cs}x + \frac{b}{-cs}y = 1.$

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As such we see that Rado's Theorem **fails** in Γ.Moreover, Γ cannot contain 3-terms arithmetic progressions.

Polyexponential equations

Fix $m \in \mathbb{N}$, $i \in [m]$ and $\boldsymbol{\alpha}_i = (\alpha_{i1}, \ldots, \alpha_{in}) \in (\mathbb{Z}^\times)^n$ such that

$$\gcd(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m) = \gcd\{\alpha_{ij} : i \in [m] \text{ and } j \in [n]\}.$$

Theorem

Given polynomials $P_1, \ldots, P_m \in \mathbb{Q}[x_1, \ldots, x_n]$, the number of solution of the equation

$$P_1(\boldsymbol{x})\boldsymbol{\alpha}_1^{\boldsymbol{x}} + \cdots + P_m(\boldsymbol{x})\boldsymbol{\alpha}_m^{\boldsymbol{x}} = 0,$$

where $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{\alpha}_i^{\boldsymbol{x}} = \alpha_{i1}^{x_1} \cdots \alpha_{in}^{x_n}$, is finite and only depends on the degree of the polynomials and the number of variables.

Polyexponential equations

Let $P_1(x,y,z) = xy - z + 2$, $P_2(x,y,z) = x - y + 2z + 2$, and $P_3(x,y,z) = xyz - z + 3$. Then the polyexponential equation

$$P_1(x,y,z)2^x3^y + P_2(x,y,z)5^x7^y + P_3(x,y,z)11^x13^y = 0$$

is not PR over $\mathbb{Z}$.

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is not PR over $\mathbb{Z}$. Indeed, fixed any $s \in \mathbb{Z}$, the equation

$$P_1(x, y, s)2^x3^y + P_2(x, y, s)5^x7^y + P_3(x, y, s)11^x13^y = 0$$

has a finite number of solutions that depend only on the degree of the polynomials and the number of variables. By the main theorem, this equation is PR iff there is a $s \in \mathbb{Z}$ such that

$$6^s(s^2 - s + 2) + 35^s(2s + 2) + 143^s(s^3 - s + 3) = 0.$$

which is impossible.

Polyexponential equations

Theorem (General case)

Given polynomials $P_1, \ldots, P_m \in \mathbb{Q}[x_1, \ldots, x_n, y]$ *The equation*

$$P_1(\boldsymbol{x}, y)\boldsymbol{\alpha}_1^{\boldsymbol{x}} + \cdots + P_m(\boldsymbol{x}, y)\boldsymbol{\alpha}_m^{\boldsymbol{x}} = 0$$

is PR over $\mathbb{Z}$ iff it admits a constant solution.

Theorem

Let $\mathcal{U} \in \beta S$ and let $f : S \to S$. Let $\overline{f} : \beta S \to \beta S$ be the continuous extension of f to βS. Then $\overline{f}(\mathcal{U}) = \mathcal{U}$ if and only if there exists $A \in \mathcal{U}$ such that $f(a) = a$ for all $a \in A$.

Lemma

Let $\mathcal{U} \in \beta S$ and let $\varphi_1(x_1, \ldots, x_n), \ldots, \varphi_k(x_1, \ldots, x_n)$ be properties on S. The following are equivalent:

1. *Given any $A \in \mathcal{U}$ there exists $a_1, \ldots, a_n \in A$ and $j \in [k]$ such that $\varphi_j(a_1, \ldots, a_n)$ is satisfied; and*
2. *there exists a $j \in [k]$ such that for all $A \in \mathcal{U}$ one can find $a_1, \ldots, a_n \in A$ satisfying $\varphi_j(a_1, \ldots, a_n)$.*

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Let S be an infinite set, let $n \in \mathbb{N}$ and let $f_1, \ldots, f_m : S^{n+1} \to S$. Let $\sigma(x_1, \ldots, x_n, x_{n+1}) = 0$ be the system of functional equations

$$\begin{cases} f_1(x_1, \ldots, x_n, x_{n+1}) &= s_1; \\ \vdots & \\ f_m(x_1, \ldots, x_n, x_{n+1}) &= s_m. \end{cases}$$

Suppose that there exists $k \in \mathbb{N}$ such that for all $s \in S$ the number of solutions in the variables $x_1, \ldots, x_n$ of $\sigma(x_1, \ldots, x_n, s) = 0$ is at most k. Then the system $\sigma(x_1, \ldots, x_n, x_{n+1}) = 0$ is partition regular on S if and only if it is trivially PR.

For all $A \subseteq S$, let

$$A' = \{s \in A : \exists s_1, \ldots, s_n \in A \text{ st } \sigma(s_1, \ldots, s_n, s) = 0\}.$$

By the hypothesis, there are

$$\psi_1, \ldots, \psi_k : S' \to S^n$$

such that whenever $s_1, \ldots, s_{n+1}$ is a solution to the system then $\exists j \in [k]$ st $\psi_j(s_{n+1}) = (s_1, \ldots, s_n)$.

Let $\mathcal{U} \in \beta S$ be a witness of the PR of this system. Then for all $A \in \mathcal{U}$ we have that $A' \in \mathcal{U}$. Thus, by the previous Lemma, TFAE

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Fix such j_0 and let

$$A'' = \{a \in A : \psi_{j_0}(a) \in A^n\}.$$

Then $A'' \in \mathcal{U}$. Let $\pi_i : S^n \to S$ be the projection onto the i-th coordinate. Then

$$A'' \subseteq A''' := \{a \in A : \pi_i \circ \psi_{j_0}(a) \in A\}$$

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Then there is a $B \in \mathcal{U}$ such that $\pi_i \circ \psi_{j_0}|_B = \text{id}$.

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Then there is a $B \in \mathcal{U}$ such that $\pi_i \circ \psi_{j_0}|_B = \mathrm{id}$.Then $B \cap B' \cap B''$ contains a constant solution to the system.

Thank you.

Thank you.

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