Donaldson–Thomas invariants, flow trees, and log Gromov–Witten invariants.

PISA Lecture Series

June 23 - 27 2025

PLAN :

Monday 1) - 2) : Donaldson-Thomas theory and wall-crossing

Tuesday 3) The flow tree formula for quiver DT invariants

Thursday 4) Quiver DT / Log GW correspondence

Friday 5) Holomorphic Floer theory

1) - 2) : Donaldson-Thomas theory and wall-crossing
(DT)

- DT invariants = virtual counts of stable objects in Calabi-Yau categories of dimension 3

- Wall-crossing : change of DT invariants when the notion of "stability" changes.

Categories.

$\mathcal{E}$: Triangulated category over $\mathbb{C}$ (+ dg-enhancement)

- E, F $\quad$ $\mathrm{Hom}(E,F)$ $\mathbb{C}$-vector space

$$\mathrm{Hom}(E,F) \otimes \mathrm{Hom}(F,G) \to \mathrm{Hom}(E,G)$$

$\mathbb{C}$-linear

- Shift $E \mapsto E[1]$

$$\mathrm{Ext}^i(E,F) \simeq \mathrm{Hom}(E, F[i])$$

<u>Example</u>: A $\mathbb{C}$-algebra

$\mathcal{E} = D^b(A\text{-mod})$ $\quad$ bounded derived category of the abelian category of A-modules.

Example: X scheme over $\mathbb{C}$ $\quad \mathcal{E} = D^b\,\mathrm{Coh}(X)$ bounded derived category of coherent sheaves on X

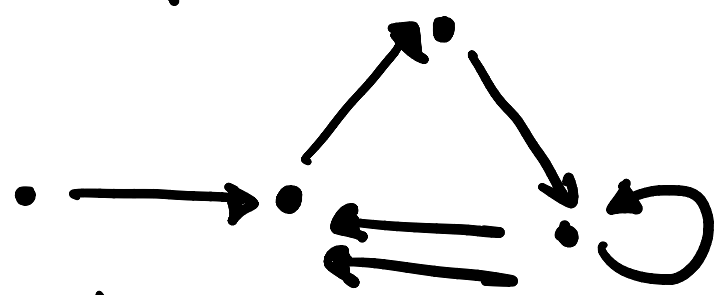
Assume $\mathcal{E}$ smooth and proper

$$\underbrace{\text{proper}}: \quad \dim_{\mathbb{C}} \bigoplus_{i \in \mathbb{Z}} \mathrm{Ext}^i(E,F) < \infty \quad \forall E,F$$

Example:
- $\mathcal{E} = D^b\,\mathrm{Coh}(X)$ $\quad X$ smooth and proper over $\mathbb{C}$
- $\mathcal{E} = D^b_Z\,\mathrm{Coh}(X)$ $\quad X$ smooth over $\mathbb{C}$, $Z \hookrightarrow X$ proper

$$= \left\{ E \in D^b\,\mathrm{Coh}(X) \;\middle|\; \mathcal{H}^i(E) \text{ set-theoretically supported on } Z \ \forall i \right\}$$

<u>Example</u>: Q = quiver = finite oriented graph



$\mathbb{C}Q$:= path-algebra of Q

$\mathbb{C}Q$-module = quiver representation

$$D^b_{fd}(\mathbb{C}Q\text{-mod}) = \left\{ V \in D^b(\mathbb{C}Q\text{-mod}) \;\middle|\; \dim_{\mathbb{C}} \bigoplus_{i\in\mathbb{Z}} H^i(V) < \infty \right\}$$

is a smooth proper category

Calabi-Yau categories.

<u>Def</u>: A smooth and proper triangulated category $\mathcal{E}$ is Calabi-Yau of dimension d if there exists isomorphisms

$$\mathrm{Ext}^i(E,F) \simeq \mathrm{Ext}^{d-i}(F,E)^* \text{ functorial in } E \text{ and } F$$

<u>Examples</u>:

1) $\mathcal{E} = D^b\mathrm{Coh}(X)$ is CY dim d — X smooth proper <u>Calabi-Yau</u> variety over $\mathbb{C}$, $\dim_{\mathbb{C}} X = d$

$K_X \simeq \mathcal{O}_X$

Serre duality

$$\mathrm{Ext}^i(E,F) \simeq \mathrm{Ext}^{d-i}(F, E\otimes K_X)^*$$
$$\simeq \mathrm{Ext}^{d-i}(F,E)^*$$

X Calabi-Yau

2) $\mathcal{E} = \mathrm{Fuk}(Y, \omega)$

(Y, ω): compact symplectic manifold $\dim_{\mathbb{R}} Y = 2d$

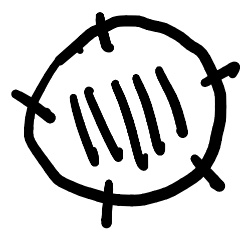
ω: Closed non-degenerate 2-form

Calabi-Yau for $\mathbb{Z}$-grading $(c_1(Y) = 0)$

$\mathcal{E}$: CY dim d

$\mathrm{Fuk}(Y,\omega)$: Fukaya category of (Y, ω)

Objects ~ Lagrangian submanifolds $L \subset Y$
$[\omega|_L = 0, \ \dim_{\mathbb{R}} L = d]$

Morphisms ~  # holomorphic disks

Cyclic symmetry $\Rightarrow$ CY property

L oriented:

$\mathrm{Ext}^i(L, L) \sim H^i(L, \mathbb{C})$

$H^i(L, \mathbb{C}) \simeq H^{d-i}(L, \mathbb{C})^*$

Poincaré duality for L

3) Q quiver $D^b_{f.d}(\mathbb{C}Q\text{-mod})$ not Calabi-Yau in general

Nevertheless:

$$(Q, W) \rightsquigarrow \mathcal{C}_{(Q,W)} = D^b_{fd}(\Gamma)$$

W: potential for Q
formal linear combination of cycles in Q

$W \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$

$\mathcal{C}_{(Q,W)}$: Calabi-Yau of dim 3

$G =$ Ginzburg dg-algebra of (Q, W)

$G = (\mathbb{C}\tilde{Q}, d)$

$\tilde{Q}$
- $a : i \to j$ deg 0 $\forall a$ arrow of Q
- $a^* : j \to i$ deg 1
- $l_i : i \to i$ deg 2 $\forall i$ vertex of Q

differential:

$$\begin{cases} da = 0 \\ da^* = \dfrac{\partial W}{\partial a} \\ dl_i = \sum_a e_i [a, a^*] e_i \end{cases}$$

e_i : "lazy" path at vertex i

$$\mathcal{C}_{(Q,W)} := D^b_{fd}(G) = \{ V \text{ dg-modules} \mid \dim_{\mathbb{C}} \bigoplus_{i \in \mathbb{Z}} H^i(V) < \infty \}$$

$$H^0(G) = \mathbb{C}Q / (\partial W) =: J_{(Q,W)} \text{ Jacobi algebra}$$

$\exists$ Heart $J_{(Q,W)}\text{-mod}_{fd} \subset \mathcal{C}$

Examples of CY categories: 1) $D^b Coh(X)$ X CY

2) $Fuk(Y, \omega)$ Y CY

3) $\mathcal{E}_{(Q,w)}$

Relations:

1) $\leftrightarrow$ 2) : Homological mirror symmetry $X \xleftrightarrow{\text{Mirror}} (Y, \omega)$

1)-2) $\leftrightarrow$ 3) : Example X smooth toric Calabi-Yau 3-fold

$\Rightarrow$ Non-compact

$$\begin{array}{c} X \\ \big\downarrow \pi \\ X^{can} = \operatorname{Spec} H^0(X, \mathcal{O}_X) \\ \uparrow \\ 0 \in \text{Affine singular} \\ \text{Toric variety} \end{array}$$

$\exists\, (Q, W)$ s.t.

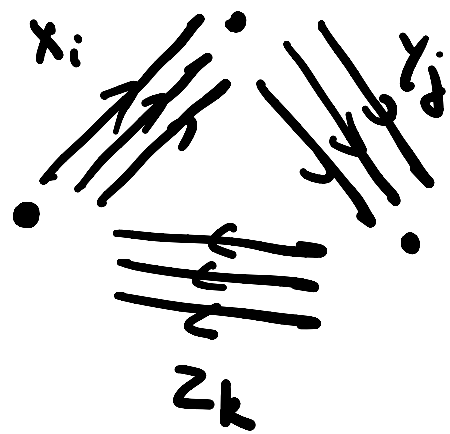
$$D^b_{\pi^{-1}(0)} Coh(X) \simeq \mathcal{E}_{(Q,W)}$$

[Dimer model]

Example:

$X = \mathbb{C}^3$ $\quad Q =$ 

$\quad W = xyz$

$X = K_{\mathbb{P}^2}$ $\quad Q =$ 

$\quad W = \sum_{i,j,k} \varepsilon_{ijk} \, x_i y_j z_k$

Stability

Definition: A stability function (= central charge) Z on an abelian category $\mathcal{A}$ linear over $\mathbb{C}$ is a group homomorphism

$$Z : K(\mathcal{A}) \longrightarrow \mathbb{C}$$

s.t $\forall E \in \mathcal{A}$, $E \neq 0$

$$Z(E) \in \mathbb{H} = \{ z = m e^{i\pi\phi} \mid m > 0, \phi \in (0,1] \}$$

$=$

Phase of E: $\phi(E) := \frac{1}{\pi} \operatorname{Arg} Z(E) \in (0,1]$

Definition: E Z-**semistable** if $\phi(A) \leq \phi(E)$ $\forall A \subset E$.

Z-**stable** if $\phi(A) < \phi(E)$ $\forall A \subsetneq E$.

Schur's lemma: E Z-stable $\Rightarrow$ $\mathrm{Hom}(E,E) = \mathbb{C}\,\mathrm{Id}$

$\mathrm{Aut}(E) = \mathbb{C}^* \mathrm{Id}$

<u>Definition</u>: A Bridgeland stability condition on a triangulated category $\mathcal{E}$ is a pair $(\mathcal{A}, Z)$, where $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{E}$, and Z is a stability function on $\mathcal{A}$ such that:

Z satisfies the Harder-Narasimhan property:

$\forall E \in \mathcal{A}$, $E \neq 0$ $\quad \exists \; 0 = E^0 \subset E^1 \subset \dots \subset E^n = E$

s.t. E^{i+1}/E^i semistable of phase ϕ_i, $\phi_1 > \dots > \phi_n$.

Fix. Γ finitely generated group

- $\omega : \Gamma \times \Gamma \to \mathbb{Z}$ bilinear form
  $\gamma_1, \gamma_2 \mapsto \omega(\gamma_1, \gamma_2)$
- $K_0(\mathcal{E}) \xrightarrow[\ell]{} \Gamma$ group homomorphism s.t.
  $\omega(\ell(E), \ell(F)) = \chi(E,F) = \sum_i (-1)^i \dim \mathrm{Ext}^i(E,F)$ Euler form

$$\mathrm{Stab}_\Gamma(\mathcal{E}) = \left\{ \begin{array}{l} \sigma = (\mathcal{A}, Z) \text{ stability conditions s.t.} \\ \quad 1) \quad Z : K_0(\mathcal{E}) \to \mathbb{C} \text{ factors as } K_0(\mathcal{E}) \xrightarrow{\ell} \Gamma \to \mathbb{C} \\ \quad 2) \ Z \text{ satisfies the support property:} \\ \forall \text{ norm } \|\cdot\| \text{ on } \Gamma \otimes \mathbb{R},\ \exists C > 0 \text{ s.t.} \\ \quad |Z(E)| \geq C\|E\| \quad \forall E\ \sigma\text{-semistable} \end{array} \right.$$

Theorem [Bridgeland]

$\mathrm{Stab}_\Gamma(\mathcal{E})$ has a natural structure of finite dimensional complex manifold such that $\forall E \in \mathcal{E} \setminus \{0\}$

$$Z_E : \mathrm{Stab}_\Gamma(\mathcal{E}) \longrightarrow \mathbb{C} \quad \text{is holomorphic.}$$
$$(\mathcal{A}, Z) \longmapsto Z(E)$$

Example: $\mathcal{E} = D^b\mathrm{Coh}(X)$ X smooth projective curve

$\mathcal{A} = \mathrm{Coh}(X)$ $\quad Z : K_0(X) \longrightarrow \mathbb{C}$

$$E \longmapsto -\deg(E) + i\,\mathrm{rk}(E)$$

$(\mathcal{A}, Z)$ is a Bridgeland stability condition on $\mathcal{E}$

Example: $\mathcal{E} = \mathcal{E}_{(Q,W)}$ (Q,W) quiver with potential

$K_0(\mathcal{E}) = \mathbb{Z}^{\#\text{vertices}} = \Gamma$

$\mathcal{A} = J_{(Q,W)}\text{-mod}$ $\forall i$ vertex fix $z_i \in \mathbb{H}$

$$\begin{aligned} Z : \Gamma &\longrightarrow \mathbb{C} \\ (n_i) &\longmapsto \sum_i n_i z_i \end{aligned}$$

Then $(\mathcal{A}, Z)$ is a Bridgeland stability condition on $\mathcal{E}$.

[King's stability]

Example: Y compact CY $\quad (Y, J)$ compact Kähler

$\dim_{\mathbb{C}} Y = d \quad K_Y \simeq \mathcal{O}_Y \quad \Omega_Y$

$\alpha \in H^2(X, \mathbb{R})$ Kähler class

Yau's theorem: $\exists!$ Ricci-flat
(Calabi conj) Kähler form ω s.t. $[\omega] = \alpha$

$$\frac{\omega^d}{d!} = (-1)^{d(d-1)/2} \left(\frac{i}{2}\right)^d \Omega_Y \wedge \overline{\Omega}_Y$$

$L \subset Y$ Lagrangian $\Omega_Y|_L = f_L \, \mathrm{vol}_L \quad f_L : L \xrightarrow[C^\infty]{} U(1)$

Definition: L special Lagrangian of phase θ if $f_L = e^{i\theta}$
(Constant function)

$$K_0(\mathrm{Fuk}(Y, \omega)) \longrightarrow \Gamma = H_d(X, \mathbb{Z})$$
$$L \longmapsto [L]$$

<u>Expectation</u> [Thomas-Yau, Joyce]

$\exists$ $(\mathcal{A}, Z)$ Bridgeland stability condition on $\mathrm{Fuk}(Y, \omega)$ such that

$$Z : \Gamma = H_d(X, \mathbb{Z}) \longrightarrow \mathbb{C}$$
$$\gamma \longmapsto \int_\gamma \Omega$$

and stable objects of phase θ ~ <u>special</u> Lagrangian submanifolds of phase θ.

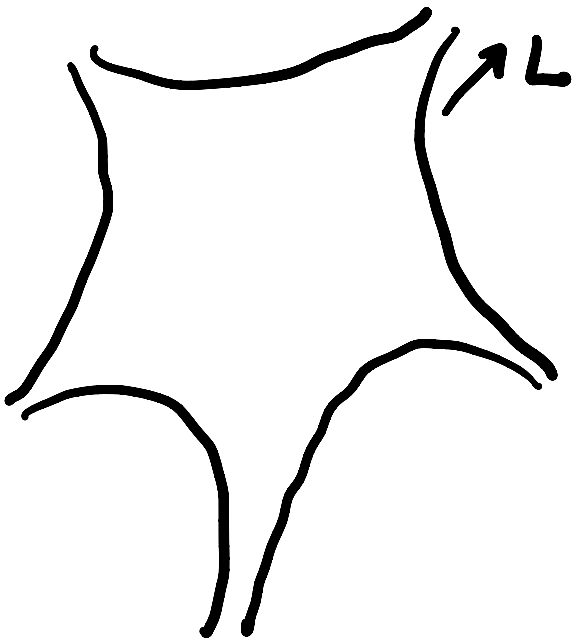
<u>Example</u>: X compact CY $K_X \simeq \mathcal{O}_X$
dim d

$$\Gamma = \mathrm{Im}(\mathrm{ch}: K_0(X) \to H^{\mathrm{even}}(X, \mathbb{Q}))$$

Expectation: $\exists \quad \widetilde{\mathcal{M}} \longrightarrow \mathrm{Stab}_\Gamma(D^b\mathrm{Coh}(X))$

"stringy Kähler moduli space"

= moduli space of complex structure on mirror s.t. near a



"large volume point" determined by an ample line bundle L,

" $\mathcal{A} \longrightarrow \mathrm{Coh}(X)$ "

" $(\mathcal{A}, Z) \longrightarrow$ Gieseker stability with respect to L "

X $\quad L$ ample line bundle $\quad E \in Coh(X)$

Reduced Hilbert polynomial $\displaystyle p_E(n) = \frac{\chi(E \otimes L^{\otimes n})}{\text{leading term}}$

<u>Definition</u>: $E \in Coh(X)$ is Gieseker (semi)stable with respect to L if:

1) E is of pure dimension

2) $\forall F \subsetneq E \quad p_F(n) < p_E(n)$ for $n \gg 0$
$(\leq)$

Moduli spaces.

$\mathcal{E}$ smooth and proper triangulated category

$\sigma = (\mathcal{A}, Z)$ Bridgeland stability condition $\gamma \in \Gamma$

$\mathfrak{M}_\gamma$ = stack of objects of class γ in $\mathcal{A}$
(derived)

$$\mathfrak{M}_\gamma^{\sigma\text{-st}} \subset \mathfrak{M}_\gamma^{\sigma\text{-sst}} \subset \mathfrak{M}_\gamma$$

stack of stable objects

stack of semistable objects

Good moduli spaces [Alper-Halpern-Leistner-Heinloth]

$$M_\gamma^{\sigma\text{-st}} \subset M_\gamma^{\sigma\text{-sst}}$$

"Coarse moduli space of stable objects"

proper scheme
"Coarse moduli space of semistable objects"

$$\begin{tikzcd} \mathfrak{M}_\gamma^{\sigma\text{-st}} \arrow[r, hook] \arrow[d, "B\mathbb{C}^*"'] & \mathfrak{M}_\gamma^{\sigma\text{-sst}} \arrow[d] \\ M_\gamma^{\sigma\text{-st}} \arrow[r, hook] & M_\gamma^{\sigma\text{-sst}} \end{tikzcd}$$

Examples : 1) Moduli spaces of quiver representations

$Q \quad \gamma = (\gamma_i) \qquad M_\gamma^{\sigma\text{-sst}} = \Big(\prod_{\alpha : i \to j} \mathbb{C}^{\gamma_i \gamma_j} \Big) /\!\!/_{GIT} \prod_i GL(\gamma_i, \mathbb{C})$

2) Moduli spaces of Gieseker semistable sheaves on smooth projective variety X.

CY3

What is special about $\mathcal{E}$ Calabi-Yau of dimension 3 ?

$M_\gamma^{\text{s-st}}$ is of expected dimension 0 :

Tangent space at E : $\mathrm{Ext}^1(E,E)$

Obstruction space at E : $\mathrm{Ext}^2(E,E)$

$$\underset{\substack{\uparrow \\ \text{CY3}}}{\simeq} \mathrm{Ext}^1(E,E)^*$$

Symmetric obstruction theory

Perfect: $\mathrm{Ext}^0(E,E) \simeq \mathrm{Ext}^3(E,E)^* = \mathbb{C}$

[Li-Tian, Behrend-Fantechi]

$\Rightarrow$ Virtual fundamental class $[M_\gamma^{\sigma\text{-st}}]^{\text{vir}}$
0-dimensional

If $M_\gamma^{\sigma\text{-st}} = M_\gamma^{\sigma\text{-sst}} \rightarrow M_\gamma^{\sigma\text{-st}}$ proper:

$$\boxed{\Omega_\gamma^\sigma := \int_{[M_\gamma^{\sigma\text{-st}}]^{\text{vir}}} 1 = \deg [M_\gamma^{\sigma\text{-st}}]^{\text{vir}} \in \mathbb{Z}}$$

DT invariant (Invariant / "deformations")

[Donaldson-Thomas 98, in the context of
Gieseker stable sheaves]

Example: X CY 3-fold

$\gamma = (1, 0, -\beta, -n)$ $\quad E = I_Z \quad Z \subset X$ 1-dim

$[Z] = \beta \quad \chi(\mathcal{O}_Z) = n$

$$0 \to I_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

$$M_\gamma^{\sigma\text{-st}} = M_\gamma^{\sigma\text{-sst}} \simeq \mathrm{Hilb}_{(\beta,n)}(X)$$

$$DT_{\beta,n} := \Omega^\sigma_{(1,0,-\beta,-n)} = \int_{[\mathrm{Hilb}_{(\beta,n)}(X)]^{\mathrm{vir}}} 1 \in \mathbb{Z}$$

MNOP conjecture:

[Pardon Thm]

$$DT_{\beta,n} \longleftrightarrow \underbrace{GW_{g,\beta}}_{\text{Gromov-Witten invariants of } X}$$

Example: If obstruction sheaf = obstruction bundle

$$\begin{array}{l} Ob \cong T^*M_\gamma^{st} \\ \downarrow \\ M_\gamma^{\sigma\text{-}st} \leftarrow \text{Smooth} \end{array}$$

$$[M_\gamma^{\sigma\text{-}st}]^{vir} = e(Ob) \cap [M_\gamma^{\sigma\text{-}st}]$$

$$\Omega_\gamma^\sigma = \int_{[M_\gamma^{\sigma\text{-}st}]} e(Ob) = \int_{[M_\gamma^{\sigma\text{-}st}]} e(T^*M_\gamma^{\sigma\text{-}st})$$

$$\Omega_\gamma^\sigma = (-1)^{\dim M_\gamma^{\sigma\text{-}st}} \int_{[M_\gamma^{\sigma\text{-}st}]} e(TM_\gamma^{\sigma\text{-}st})$$

$$\boxed{\Omega_\gamma^\sigma = (-1)^{\dim M_\gamma^{\sigma\text{-}st}} \chi(M_\gamma^{\sigma\text{-}st})}$$

What if $M_\gamma^{\sigma\text{-st}} \neq M_\gamma^{\sigma\text{-sst}}$? [Joyce, Joyce-Song 2008; Kontsevich-Soibelman]

How to count semistable objects? [Reineke-Meinhardt, 2014; Davison-Meinhardt, ...]

<u>Example:</u> $Q = \bullet \quad \gamma = n$

$$\mathbb{C}^n \quad \mathfrak{M}_\gamma^{\sigma\text{-sst}} = \frac{\operatorname{Spec}\mathbb{C}}{GL(n,\mathbb{C})} = BGL(n,\mathbb{C})$$

$$\chi(\mathfrak{M}_\gamma^{\sigma\text{-sst}}) = \frac{1}{\chi(GL(n,\mathbb{C}))} = \frac{1}{0} = ?$$

Motivic or cohomological counts make sense:

$$|\mathfrak{M}_\gamma^{\sigma\text{-sst}}(\mathbb{F}_q)| = \frac{1}{|GL(n,\mathbb{F}_q)|} = \frac{1}{(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})}$$

$$H^\bullet(BGL(n,\mathbb{C}))$$

Problem: How to have a well-defined limit $q \to 1$?

Take a logarithm. Key identity:

$$\sum_{n \geq 0} \frac{(-q^{1/2})^{n^2}}{(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})} x^n = \exp\left(-\sum_{n\geq 1} \frac{1}{n} \frac{1}{q^{\frac{n}{2}} - q^{-\frac{n}{2}}} x^n\right)$$

$$\parallel$$

$\psi(x)$ quantum dilogarithm

Categorification (with correct shifts and twists):

$$\bigoplus_n H^*(BGL_n(\mathbb{C})) = \mathrm{Sym}\left(H^*(B\mathbb{C}^*)\right)$$

$$q = e^{\hbar}$$

$$\psi(x) \underset{\hbar \to 0}{\sim} e^{-\sum_n \frac{x^n}{n^2}}$$

$$\sim e^{-\mathrm{Li}_2(x)}$$

Virtual motivic or cohomological count?

$\mathfrak{M}_\gamma^{\text{б-sst}}$ is locally the critical locus of a function defined on a smooth stack!

Formal locally:

E Deformation theory $\mathrm{Ext}^\bullet(E,E)$ A_∞-algebra

$$m_n : \mathrm{Ext}^1(E,E)^{\otimes n} \to \mathrm{Ext}^2(E,E) \qquad n \geq 2$$

Locally $\mathfrak{M}_\gamma = m^{-1}(0) \subset \mathrm{Ext}^1(E,E)$

$$m : \mathrm{Ext}^1(E,E) \to \mathrm{Ext}^2(E,E)$$

$$a \mapsto \sum_n m_n(a,\dots,a)$$

[Maurer-Cartan equation:

$$da + a^2 = 0$$

$$m_1(a) + m_2(a,a) = 0 \;]$$

$\mathcal{E}$ CY3 $\quad m = dF \quad F : \mathrm{Ext}^1(E,E) \longrightarrow \mathbb{C}$

$$a \longmapsto \sum_{n \geq 1} \frac{1}{n+1} (m_n(a, \dots, a), a)$$

Serre duality pairing:

$$(-,-) : \mathrm{Ext}^1(E,E) \times \mathrm{Ext}^2(E,E) \to \mathbb{C}$$

[Chern-Simons functional

$\frac{1}{2}(m_1(a), a) + \frac{1}{3}(m_2(a), a)$

$= \int \left(\frac{1}{2} a da + \frac{1}{3} a^3 \right)$]

Zariski locally:

- (-1)-shifted symplectic structure [Pantev-Toen-Vaquie-Vezzosi]
- Darboux theorem [Ben-Bassat, Brav, Bussi-Joyce]

Locally $\mathcal{M}_\gamma^{\sigma\text{-sst}} = \mathrm{Crit}(F : \tilde{\mathcal{M}} \to \mathbb{C})$

↑ smooth

↳ φ_F perverse sheaf of vanishing cycles

Global choice (orientation data $= K_{\mathcal{M}}^{\mathrm{vir},1/2}$)

↳ Glue to a global perverse sheaf φ on $\mathcal{M}_\gamma^{\sigma\text{-st}}$

Fix phase θ, define $\Omega_\gamma(q)$ by:

$$\sum_{\substack{\gamma \\ \mathrm{Arg}\, Z_\gamma = \theta}} \chi_q(\mathcal{M}_\gamma^{\sigma\text{-sst}}, \varphi)\, x^\gamma = \exp\left(\sum_{\substack{\gamma \\ \mathrm{Arg}\, Z_\gamma = \theta}} \sum_{n \geq 1} -\frac{1}{n} \frac{\Omega_\gamma^\sigma(q^n)}{q^{\frac{n}{2}} - q^{-\frac{n}{2}}} x^{n\gamma} \right)$$

Symmetrized Weight polynomial

A priori: $\Omega^{\sigma}_{\gamma}(q) \in \mathbb{Q}(q^{\pm\frac{1}{2}})$

Integrality conjecture [Joyce, Kontsevich-Soibelman]:

for "generic" σ: $\boxed{\Omega^{\sigma}_{\gamma}(q) \in \mathbb{Z}[q^{\pm 1/2}]}$ Refined/Motivic DT invariant

Can set $q=1$:

$\boxed{\Omega^{\sigma}_{\gamma} \in \mathbb{Z}}$ DT invariant

Example: (Q, W)

[Meinhardt–Reineke]

[Davison–Meinhardt]

$$\mathcal{M}_\gamma = \mathrm{Crit}\big(\mathrm{Tr}(W) : \underbrace{\mathrm{Rep}_\gamma(Q)}_{\text{smooth stack}} \longrightarrow \mathbb{C}\big)$$

Global description as a critical locus (no need for orientation data)

$$\Omega^\sigma_\gamma(q) = \chi_q\Big(H^*\big(\mathrm{Rep}^{\sigma\text{-sst}}_\gamma, \Phi_{\mathrm{Tr}(W)}(j_{!*}\mathbb{Q})\big)\Big) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$$

Coarse moduli space

$j : \mathrm{Rep}^{\sigma\text{-st}}_\gamma \hookrightarrow \mathrm{Rep}^{\sigma\text{-sst}}_\gamma$

$\Phi_{\mathrm{Tr}(W)}$: vanishing cycle functor

$$\Omega^\sigma_\gamma = \Omega^\sigma_\gamma(q=1)$$

Remark: BPS Lie algebra

E.g [Maria-Botta, Davison]

Maulik-Okounkov Lie algebra of a quiver Q $=$ BPS Lie algebra of tripled quiver $(\tilde{Q}, \tilde{W})$

$(\curvearrowright QH^*(\text{Nakajima quiver varieties}))$

SUMMARY:

INPUT: • $\mathcal{C}$: smooth proper CY3 (+ orientation data)

• $\sigma = (\mathcal{A}, Z)$ stability condition

• $\gamma \in \Gamma$

OUTPUT:

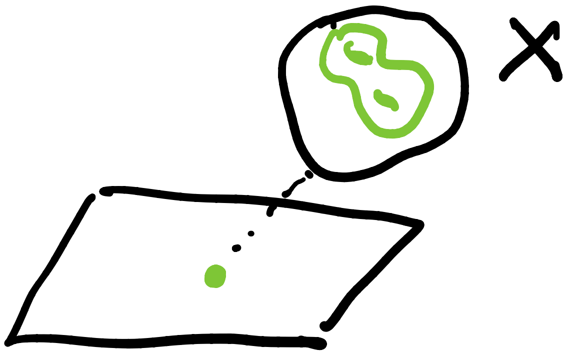
• DT invariants $\Omega^{\sigma}_{\gamma} \in \mathbb{Z}$

Virtual counts of σ-stable objects of class γ in $\mathcal{C}$.

PHYSICS INTERPRETATION

X CY 3-fold IIA string theory on $\mathbb{R}^4 \times X$

D0-D2-D4-D6 branes on X $\longrightarrow$ BPS particles in $\mathbb{R}^4$



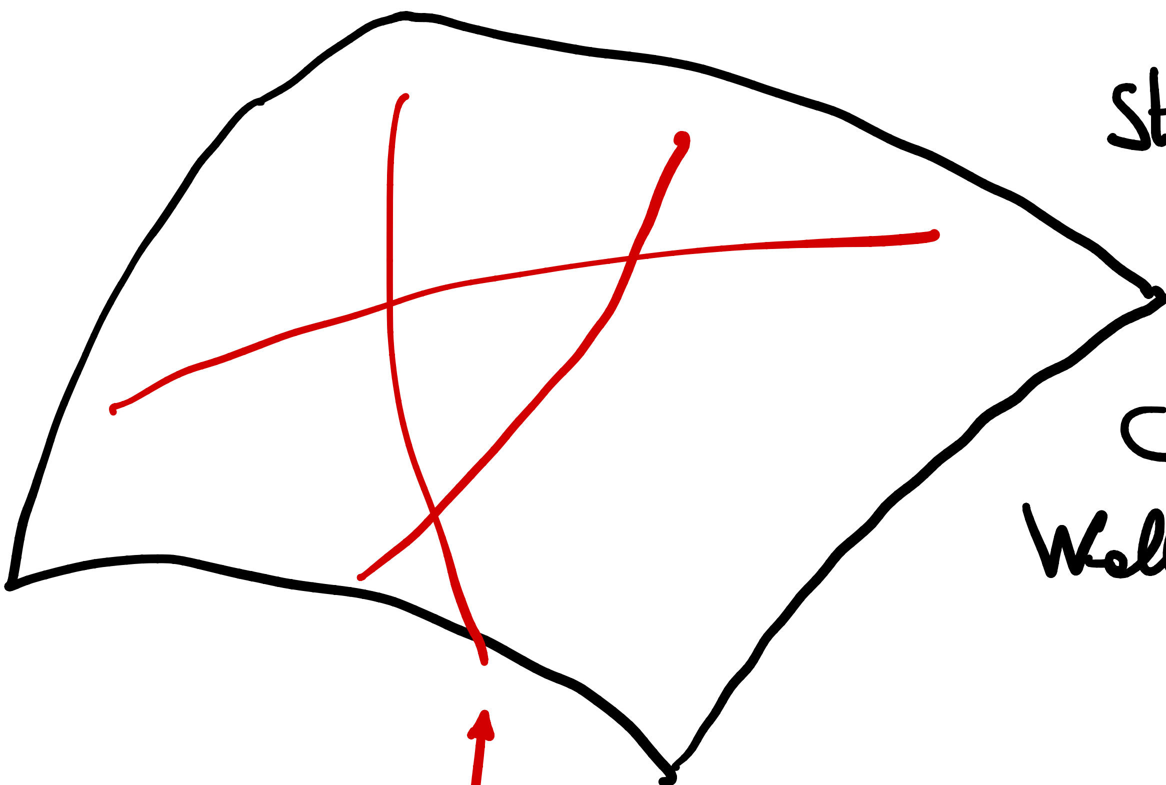
Y CY 3-fold IIB string theory on $\mathbb{R}^4 \times Y$

D3-branes on X $\longrightarrow$ BPS particles in $\mathbb{R}^4$

Ω_γ^6 = # BPS particles / Black holes of electro-magnetic charge γ.

WALL-CROSSING

$\Omega_\gamma^\sigma \leftarrow \sigma \in \mathrm{Stab}_\Gamma(\mathcal{E})$



$\mathrm{Stab}_\Gamma(\mathcal{E})$

Chamber and Walls structures.

Walls $\{\sigma \mid Z^\sigma_{\gamma_1} /\!/ Z^\sigma_{\gamma_2},\ \gamma_1 \nparallel \gamma_2\}$

WALL CROSSING FORMULA

Γ $\quad \sigma = (\mathcal{A}, Z)$ $\quad Z : \Gamma \to \mathbb{C}$

$\gamma \mapsto Z_\gamma$

$$K_0(\mathcal{E}) \xrightarrow{cl} \Gamma$$

$$\chi(E,F) = \sum_{i=0}^{3} (-1)^i \dim \mathrm{Ext}^i(E,F)$$

Euler form $\quad$ CY3 $\Rightarrow$ Skew-symmetric $\chi(E,F) = -\chi(F,E)$

$$\omega : \Gamma \times \Gamma \to \mathbb{Z}$$
$$(\gamma_1, \gamma_2) \mapsto \omega(\gamma_1, \gamma_2)$$

Skew-symmetric s.t.

$$\omega(cl(E), cl(F)) = \chi(E,F)$$

$$T = \mathrm{Spec} \langle X^\gamma, \gamma \in \Gamma \mid X^{\gamma_1} X^{\gamma_2} = (-1)^{\omega(\gamma_1,\gamma_2)} X^{\gamma_1+\gamma_2} \rangle$$

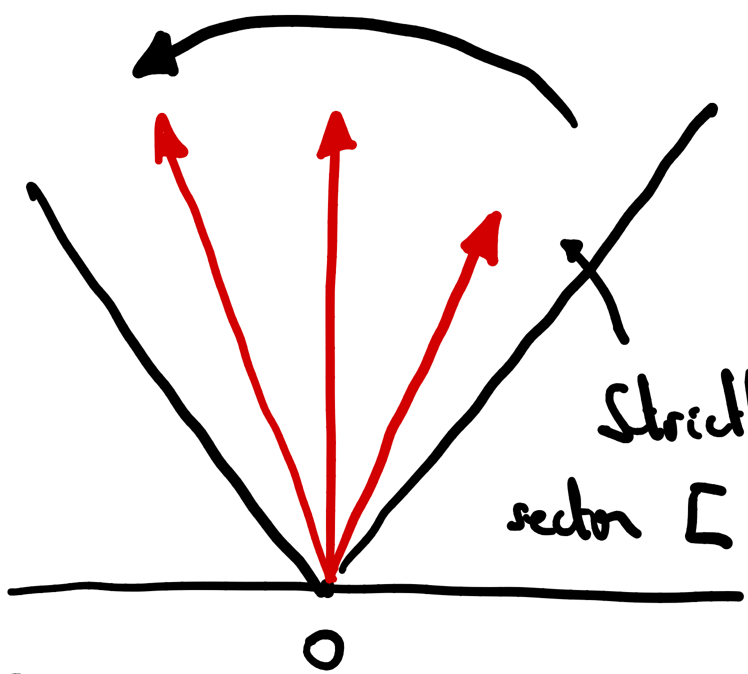
Torsor / $\mathrm{Spec}\, \mathbb{C}[\Gamma]$

Poisson bracket $\{X^{\gamma_1}, X^{\gamma_2}\} := (-1)^{\omega(\gamma_1,\gamma_2)} \omega(\gamma_1,\gamma_2) X^{\gamma_1+\gamma_2}$

$$K_\gamma = \exp(\{-\mathrm{Li}_2(X_\gamma), -\})$$ Poisson automorphism

$$\mathrm{Li}_2(X_\gamma) = \sum_{n \geq 1} \frac{X_\gamma^n}{n^2}$$ Hamiltonian

$$K_\gamma : X_{\gamma'} \longmapsto (1 - X_\gamma)^{\omega(\gamma,\gamma')} X_{\gamma'}$$



Strictly convex sector Σ in $\mathbb{C}$

WALL CROSSING FORMULA:

$$\overleftarrow{\prod_{\substack{\gamma \\ Z_\gamma \in \Sigma}}} K_\gamma^{\Omega_\gamma} = \mathrm{Cst}$$

as long as no ray $Z_\gamma \mathbb{R}_{\geq 0}$ with $\Omega_\gamma \neq 0$ goes outside Σ

[Kontsevich-Soibelman, Joyce-Song]