# *Logarithmic Gromov-Witten invariants and its application to mirror symmetry*

M/T **I. Geometric introduction to logarithmic geometry**

T/W **II. Tropicalization and log smooth curves**

W/TH **III. Logarithmic Gromov-Witten invariants, Artin fans, punctured Gromov-Witten invariants**

TH **IV. The gluing formalism for log Gromov-Witten theory**

F **V. Intrinsic mirror symmetry**

# I. Geometric Introduction to Log Geometry

**Category:** algebraic varieties, complex analytic spaces over $\mathbb{C}$

$\quad\cap \qquad\qquad\qquad\qquad \cap$

algebraic stacks $\qquad$ analytic stacks

---

## 1. Normal crossing varieties $\qquad X$ smooth, $D \subseteq X$

- simple normal crossings (snc): $\exists\ X \supseteq \mathcal{U} \xrightarrow{\pi} \mathbb{A}^r$ smooth, $\mathcal{U} \cap D = \pi^{-1}(V(z_1 \cdots z_r))$

  $\Rightarrow D \cap \mathcal{U} = D_1 \cup \dots \cup D_r$, $\quad D_i = \pi^{-1}(V(z_i))$ smooth divisor

- generally: $\mathcal{U} \to X$ étale nbd., e.g. nodal cubic:


$D$ snc $\Rightarrow$ $X$ comes with line bundles $L_i$ with sections $s_i$, $D_i = (s_i = 0)$

$\mathcal{L}_i =$ (sheaf of regular sections of $L_i$) $= \mathcal{O}_X(D_i)$, $\quad s_i : \mathcal{O}_X \to \mathcal{O}_X(D_i)$.

$\mathcal{L}_i|_{D_i} = \mathcal{O}_{D_i}(D_i) = \mathcal{O}(N_{D_i|X})$ $\quad$ normal bundle: $N_{D_i|X} = (T_X|_{D_i})/T_{D_i}$.

dually $\quad \mathcal{L}_i^* = \mathcal{O}_X(-D_i) \hookrightarrow \mathcal{O}_X$.

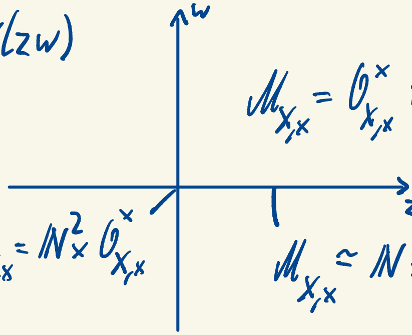
**Iterate:** $D_i \cap D_j \subseteq D_i$ snc divisor $\rightsquigarrow N_{D_i \cap D_j | D_i} = N_{D_j|X}|_{D_j}$

$$N_{D_i \cap D_j | X} = N_{D_i|X}|_{D_i \cap D_j} \otimes N_{D_j|X}|_{D_j \cap D_j}$$

All contained in $\boxed{\mathcal{M}_X := \mathcal{O}^\times_{X \setminus D} \cap \mathcal{O}_X \xrightarrow{\alpha} (\mathcal{O}_X, \cdot)}$ "log structure"

sheaf of multiplicative monoids: $f, g \in \mathcal{M}_X(\mathcal{U}) \Rightarrow f \cdot g \in \mathcal{M}_X(\mathcal{U})$

**Expl:** $X = \mathbb{A}^2$, $D = V(zw)$




$M_{X,x} = \mathcal{O}^\times_{X,x} \ni h$

$z^a w^b \cdot h$

$\Updownarrow$

$((a,b),h) \in M_{X,x} = \mathbb{N}^2 \times \mathcal{O}^\times_{X,x}$

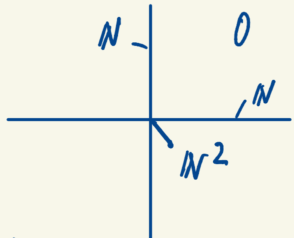
$M_{X,x} \simeq \mathbb{N} \times \mathcal{O}^\times_{X,x}$ $\quad (b,h) \leftrightarrow w^b \cdot h$

$M_X^{gp}$: associated abelian sheaf (replaces $\mathbb{N}$-factors by $\mathbb{Z}$)

**discrete part:** $\overline{M}_X := M_X / \mathcal{O}_X^\times \overset{\text{snc}}{\cong} \bigoplus \mathbb{N}_{D_i}$

$\cap$

$\overline{M}_X^{gp}$ constructible (abelian) sheaf




**Retrieval of line bundles:** $\Gamma(\overline{M}_X^{gp}) = \mathbb{Z}^r \ni (a_1, \dots, a_r)$

$\kappa: M_X^{gp} \xrightarrow{/\mathcal{O}_X^\times} \overline{M}_X^{gp}$

$f \longmapsto (\mathrm{ord}_{D_1} f, \dots, \mathrm{ord}_{D_r} f)$

$\Rightarrow T = \kappa^{-1}(a_1, \dots, a_r)$ is an $\mathcal{O}_X^\times$-torsor

assoc. invertible $\mathcal{O}_X$-module:

$(T \oplus \mathcal{O}_X)/\mathcal{O}_X^\times = \mathcal{O}_X(-\sum_i a_i D_i)$

# 2. Toric Geometries

$P \subseteq \mathbb{Z}^n = M$ finitely generated monoid, i.e. $P = \mathbb{N}m_1 + \dots + \mathbb{N}m_s \subseteq \mathbb{Z}^n$

E.g. $P = \sigma^\vee \cap \mathbb{Z}^n$, $\sigma \subseteq N_\mathbb{R} \simeq \mathbb{R}^n$ (convex) rational polyhedral cone

$$N = \mathrm{Hom}(M, \mathbb{Z}),\quad \sigma^\vee = \mathrm{Hom}_{\mathrm{Mon}}(\sigma, \mathbb{R}_{\geq 0})$$

$\rightsquigarrow$ f.g. $\mathbb{C}$-algebra $\mathbb{C}[P] = \{ \sum_{\text{finite}} a_m \chi^m \mid m = (m_1, \dots, m_n) \in P,\ a_m \in \mathbb{C} \}$, $\chi^m = \chi_1^{m_1} \cdots \chi_n^{m_n}$

$$\subseteq \mathbb{C}[M] = \mathbb{C}[\chi_1^{\pm 1}, \dots, \chi_n^{\pm 1}]$$

Generators $m_1, \dots, m_s$ define $\varphi : \mathbb{C}[u_1, \dots, u_s] \twoheadrightarrow \mathbb{C}[P]$, $\varphi(u_i) = \chi^{m_i}$

$$\Rightarrow \mathbb{C}[P] \simeq \mathbb{C}[u_1, \dots, u_s] / \ker \varphi$$

provides embedding $\mathrm{Spec}\, \mathbb{C}[P] \xrightarrow{\simeq} V(\ker(\varphi)) \subseteq \mathbb{A}^s$

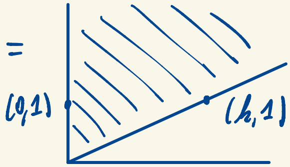
<u>Computation:</u> $\ker(\varphi)$ is a <u>toric ideal</u>, i.e. generated by binomials

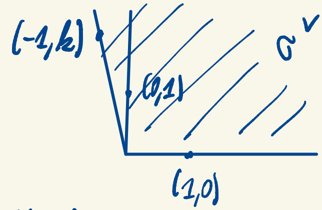
$$u_1^{a_1} \cdots u_s^{a_s} - u_1^{b_1} \cdots u_s^{b_s} \quad \forall\ \textstyle\sum_i a_i m_i = \sum_i b_i m_i \text{ in } P$$

$\mathbb{N} \supseteq \mathbb{N}\cdot 2 + \mathbb{N}\cdot 3$

<u>Fact:</u> Always $P \subseteq \sigma^\vee \cap \mathbb{Z}^n$ with $\sigma = P_\mathbb{R}^\vee = \mathrm{Hom}_{\mathrm{mon}}(P, \mathbb{R}_{\geq 0}^{+})$

with "=" $\Leftrightarrow$ $P$ is <u>saturated</u>, i.e. $dm \in P$ for $m \in \mathbb{Z}^n$, $d \in \mathbb{N} \setminus \{0\}$ $\Rightarrow m \in P$

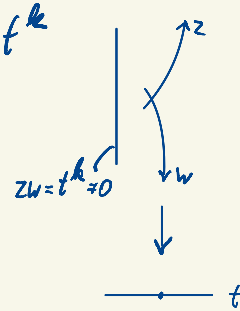
<u>Expl:</u> a) $\sigma =$ 

$\subseteq N_\mathbb{R}$ $\Rightarrow$ 

$\sigma^\vee \subseteq M_\mathbb{R}$

$\mathbb{C}[P] \simeq \mathbb{C}[z, w, t]/(zw - t^k) \Leftarrow P = \sigma^\vee \cap \mathbb{Z}^2 = \mathbb{N}\cdot(1,0) + \mathbb{N}\cdot(0,1) + \mathbb{N}\cdot(-1,k)$

$z = \chi^{(1,0)},\ w = \chi^{(-1,k)},\ t = \chi^{(0,1)}$ : $zw = \chi^{(0,k)} = t^k$

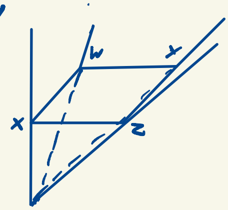



$\underline{k=1}$: semistable degeneration of $\mathbb{C}^*$ to $\mathbb{C} \amalg_0 \mathbb{C}$

$\underline{k>1}$: base change by $t \mapsto t^k$

b) $\sigma^\vee$ = cone over square $\operatorname{conv}\{(0,0),(1,0),(1,1),(0,1)\}$ $[\sigma = \langle (1,0,0),(0,1,0),(-1,0,1),(0,-1,1)\rangle]$

$= \mathbb{R}_{\geq 0}\cdot(0,0,1) + \mathbb{R}_{\geq 0}\cdot(1,0,1) + \mathbb{R}_{\geq 0}\cdot(1,1,1) + \mathbb{R}_{\geq 0}(0,1,1)$

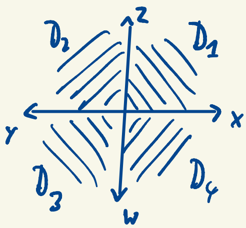

$\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^3] \simeq \mathbb{C}[x,y,z,w]/(xy-zw) \simeq \mathbb{C}[x,y] \otimes_{\mathbb{C}[t]} \mathbb{C}[z,w]$

is a $\mathbb{C}[t]$-algebra, $t \mapsto xy = zw$ $\qquad xy \leftarrow\!\shortmid\ t \mapsto zw$

$\Rightarrow \operatorname{Spec}\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^3] \simeq \mathbb{A}^2 \times_{\mathbb{A}^1} \mathbb{A}^2$ is a degeneration of $(\mathbb{C}^*)^2$ to $\bigcup_4 \mathbb{A}^2$

$$\begin{array}{c} \parallel \\ X \xrightarrow{\ \pi\ } \mathbb{A}^1 \end{array}$$

$\pi^{-1}(0) =$




$$\begin{tikzcd} \mathbb{A}^2 \arrow[dr] & & \mathbb{A}^2 \arrow[dl] \\ & \mathbb{A}^1 & \end{tikzcd}$$

$(z,w) \mapsto zw$, $\quad (x,y) \mapsto xy$

---

$\underline{\text{Torus action}}$: $P = \sigma^\vee \cap M \rightsquigarrow \mathbb{C}[P] \hookrightarrow \mathbb{C}[M]$ induces an $M$-grading on $\mathbb{C}[P]$

$\rightsquigarrow (\mathbb{C}^*)^n$-equivariant embedding $\mathbb{G}_m^n = \operatorname{Spec}\mathbb{C}[M] \hookrightarrow TV(\sigma) := \operatorname{Spec}\mathbb{C}[P]$ $\quad$ ($\mathbb{G}_m^n \circlearrowright TV(\sigma)$)

$\underline{\text{Def}}$: $\underline{\text{toric variety}}$ = $\mathbb{G}_m^n$-equvt partial compactification of $\mathbb{G}_m^n$ $\quad$ [often asked to be normal $\Leftrightarrow$ $P$ saturated]

$\quad\quad \cup\!\shortmid$

$\underline{\text{toric divisor}}$ $\quad D = TV(\sigma) \setminus \mathbb{G}_m^n = D_1 \cup \ldots \cup D_r$

$\quad\quad$ affine case $\qquad r = |\sigma(1)|$ rays of $\sigma$

Quotient construction [via Cox rings] : $N \simeq \mathbb{Z}^n$

$\sigma(1)$ = set of rays of $\sigma = \{ s_1, \dots, s_r \}$

Cox ring : $R = \mathbb{C}[\mathrm{Map}(\sigma(1), N)] = \mathbb{C}[X_1, \dots, X_r]$, $X_i = X^{e_i}$, $e_i(s_j) = \delta_{ij}$

$\sigma \subseteq N_{\mathbb{R}} \rightsquigarrow$ hom $M = \mathrm{Hom}(N, \mathbb{Z}) \overset{\simeq \mathbb{Z}^n}{\longrightarrow} \mathrm{Map}(\sigma(1), \mathbb{Z})^{\simeq \mathbb{Z}^r}$, $m \mapsto (s_i \mapsto \langle m, n_i \rangle)$

$s_i = \mathbb{R}_{\geq 0} n_i$, $n_i \in N$ primitive

Grade $R$ by $\Gamma := \mathrm{Map}(\sigma(1), \mathbb{Z})/M$.

Grading defines an action of $\mathrm{Spec}\, \mathbb{C}[\Gamma] \simeq \mathbb{G}_m^{r-n}$ on $\mathrm{Spec}\, R \simeq \mathbb{A}^r$.

categorical quotient: $\mathrm{Spec}\, R^\Gamma$, $R^\Gamma \subseteq R$ $\deg = 0$ (= inv.) subring

Fact: $R^\Gamma = \mathbb{C}[P]$, $P = \sigma^\vee \cap M$

Summary:

affine toric variety $\boxed{TV(\sigma) = \mathrm{Spec}\, \mathbb{C}[P] \simeq \mathbb{A}^r /\!/ \mathbb{G}_m^{r-n}}$

Toric log structures: $X = TV(\sigma)$

Generalizing $\mathcal{M}_{\mathbb{A}^n} = \mathcal{O}^\times_{\mathbb{A}^n | V(z_1 \cdots z_n)} \cap \mathcal{O}_{\mathbb{A}^n} \hookrightarrow \mathcal{O}_{\mathbb{A}^n}$, we obtain

$$\mathcal{M}_X = \mathcal{O}^\times_{X|D} \cap \mathcal{O}_X \longhookrightarrow \mathcal{O}_X$$

$\overline{\mathcal{M}}_X = \mathcal{M}_X / \mathcal{O}_X^\times$. Now $\Gamma(\overline{\mathcal{M}}_X^{gp}) \hookrightarrow \mathbb{Z}^r$, $f \mapsto (\mathrm{ord}_{D_1} f, \dots, \mathrm{ord}_{D_r} f)$

image $\overset{\text{eff.}}{=}$ principal (= Cartier) divisors supported on $D \simeq P$

Expl: $\sigma^\vee = \mathrm{cone}(\square)$, then $D = \sum_i a_i D_i$ Cartier $\iff a_1 + a_3 = a_2 + a_4$

# 3. Abstract log structures

$$\boxed{\alpha: \mathcal{M}_X \to (\mathcal{O}_X, \cdot) \text{ s.th. } \alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^* \text{ iso}}$$

log space $(X, \mathcal{M}_X)$

provides $\overline{\mathcal{M}}_X = \mathcal{M}_X / \mathcal{O}_X^*$, $\kappa: \mathcal{M}_X \to \overline{\mathcal{M}}_X$

hence $\forall\, \mathcal{U} \subseteq X$, $\bar m \in \Gamma(\mathcal{U}, \overline{\mathcal{M}}_X) \rightsquigarrow \kappa^{-1}(\bar m)$ $\mathcal{O}_{\mathcal{U}}^*$-torsor & $\kappa^{-1}(\bar m) \to \mathcal{O}_{\mathcal{U}}$

$$\begin{tikzcd} & \mathcal{L}_m \arrow[dr, dashed] & \\ \kappa^{-1}(\bar m) \arrow[ur, hook] \arrow[rr] & & \mathcal{O}_{\mathcal{U}} \end{tikzcd}$$

---

<u>Alternative point of view</u> (Deligne-Faltings): A log str. on $X$ is:

- a sheaf of (f.g.) monoids $\overline{\mathcal{M}}$
- $\forall\, \mathcal{U} \subseteq X$: $\overline{\mathcal{M}}(\mathcal{U}) \longrightarrow Div(\mathcal{U}) = \{(\text{line bdl, section})\}$

  $\bar m \longmapsto (\alpha_{\bar m}: \kappa^{-1}(\bar m) \to \mathcal{O}_{\mathcal{U}})^\vee = (\mathcal{L}_m^*, \alpha_m^\vee)$
- compatibility, i.e. "$\overline{\mathcal{M}} \to Div_X$" is a "symmetric monoidal functor"

e.g. $\bar m_1, \bar m_2 \in \overline{\mathcal{M}}(\mathcal{U}) \longleftrightarrow (\mathcal{L}(m_1+m_2), s_{m_1+m_2}) \overset{!}{=} (\mathcal{L}(m_1), s_{m_1}) \otimes (\mathcal{L}(m_2), s_{m_2})$

<u>Lit</u>: Borne/Vistoli "Parabolic Sheaves"

---

<u>Expl</u>: a) <u>Log points</u>: $Q$ monoid with $Q^\times = \{0\}$

$$X = \mathrm{Spec}(Q \to \mathbb{C}) := (\mathrm{Spec}\,\mathbb{C}, \underbrace{Q \oplus \mathbb{C}^*}_{=\mathcal{M}_X})$$

$0 \longmapsto 1$

$0 \neq q \longmapsto 0$

b) <u>Pull-back log str.</u>: $X \xrightarrow{f} Y$, $\mathcal{M}_Y$ log str. on $Y$

$\rightsquigarrow$ log str. on $X$: $f^*\mathcal{M}_Y := f^{-1}\mathcal{M}_Y \oplus_{f^{-1}\mathcal{O}_Y^*} \mathcal{O}_X^* \xrightarrow{\alpha_X} \mathcal{O}_X$, $(s,h) \longmapsto h \cdot (f^*\alpha)(s)$

fact: $\overline{f^*\mathcal{M}_Y} = f^{-1}\overline{\mathcal{M}}_X$

$$\begin{tikzcd} f^{-1}\mathcal{O}_Y^* \arrow[r] \arrow[d, "f^{-1}\alpha"] & \mathcal{O}_X^* \arrow[d] \\ f^{-1}\mathcal{M}_Y \arrow[r] & f^{-1}\mathcal{M}_Y \oplus_{f^{-1}\mathcal{O}_Y^*} \mathcal{O}_X^* \end{tikzcd}$$

$= (f^{-1}\mathcal{M}_Y \oplus \mathcal{O}_X^*) / f^{-1}\mathcal{O}_Y^*$

$g \cdot (s,h) = (gs, g^{-1}h)$ $\quad g$ $\quad h$

**Note**: $Y = TV(\sigma)$, $f: X = pt \hookrightarrow Y$ inclusion of 0d torus orbit

yields $(X, f^* \mathcal{M}_Y) = \mathrm{Spec}(Q \to \mathbb{C})$, $Q = \sigma^\vee \cap M$

---

**Chart for $(X, \mathcal{M}_X)$** on $U \subseteq X$: $U \xrightarrow{f} \mathrm{Spec}\,\mathbb{C}[P]$ & iso $\mathcal{M}_X|_U \simeq f^* \mathcal{M}_{\mathrm{Spec}\,\mathbb{C}[P]}$

$P \to \Gamma(U, \mathcal{O}_X)$

(alg. geom: usually in étale topology)

**fine log structure**: $\iff$ $\exists$ charts locally

**fine saturated (fs-) log str**: may choose $P$ saturated, i.e. $\simeq \sigma^\vee \cap M$

**log morphisms**: $f: (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$

means $\underline{f}: X \to Y$ $(\leadsto f^\#: f^{-1}\mathcal{O}_Y \to \mathcal{O}_X)$ &

$$\begin{tikzcd} f^{-1}\mathcal{M}_Y \arrow[r, "f^\flat"] \arrow[d, "f^{-1}\alpha_Y"'] & \mathcal{M}_X \arrow[d, "\alpha_X"] \\ f^{-1}\mathcal{O}_Y \arrow[r, "f^\#"] & \mathcal{O}_X \end{tikzcd}$$

commutative

**strict morphisms**: $f^*\mathcal{M}_Y \to \mathcal{M}_X$ iso $(\iff f^{-1}\overline{\mathcal{M}}_Y \to \overline{\mathcal{M}}_X$ iso$)$

these can be viewed as "logarithmically trivial"

## 4. Log smooth morphisms

"locally given by toric morphism",
i.e. étale locally on $X$

$$\begin{tikzcd} X \arrow[r, "\text{strict}"', "\text{(i.e. a chart)}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \operatorname{Spec} \mathbb{C}[P] \arrow[d] & P \\ Y \arrow[r, "\text{strict}"'] & \operatorname{Spec} \mathbb{C}[Q] & Q \arrow[u] \end{tikzcd}$$

Expls: a) $Q = 0 \iff X$ is toroidal, i.e. $(X, D)$ locally isomorphic to (toric var., toric div.), $D = \operatorname{supp} \overline{\mathcal{M}}^{gp}$.

b) $Q = \mathbb{N} \xrightarrow{\text{diag}} P = \mathbb{N}^r$ : nc degeneration
$1 \longmapsto (1, \dots, 1)$

c) Restriction of nc degeneration to central fiber:

$$\begin{tikzcd} \bigcup D_i = D = X_0 \arrow[r, hook] \arrow[d] & X \arrow[d] \\ 0 \arrow[r, hook] & S \text{ curve} \end{tikzcd} \quad \overset{\log}{\rightsquigarrow} \quad \begin{tikzcd} (X_0, \mathcal{M}_{X_0} = \mathcal{M}_X|_{X_0}) \arrow[r] \arrow[d] & (X, \mathcal{M}_{(X,D)}) \arrow[d] \\ \operatorname{Spec}(\mathbb{N} \to \mathbb{C}) \arrow[r] & (S, \mathcal{M}_{(S,0)}) \end{tikzcd}$$

strict

standard log point

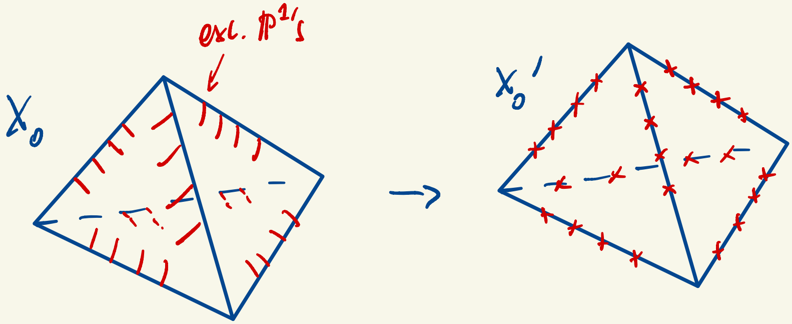
<u>Explicit expl:</u> $f \in \mathbb{C}[z_0, \dots, z_3]$ homog., $\deg = 4$, general

$$\begin{tikzcd} \mathbb{P}^3 \supseteq \bigcup_4 \mathbb{P}^2 = V(z_0 \cdots z_3) = X_0' \arrow[r, phantom, "\subseteq"] \arrow[d] & X' = V(tf + z_0 \cdots z_3) \subseteq \mathbb{P}^3 \times \mathbb{A}^2 \arrow[d] \\ 0 \arrow[r, hook] & S = \mathbb{A}^1_t \end{tikzcd}$$

but $X'$ not smooth: $X'_{sing} = (X_0)_{sing} \cap V(f) = 24 \; A_1\text{-sing's}$ ($\overset{=}{} 6 \cdot 4$)

[locally $\simeq V(xy - zw)$]

resolve:
‖
blow up $X'$
in irred. cmps
of $X_0'$ in some
order




Then
$$\begin{tikzcd} X \arrow[d] \arrow[dr] & \\ S & X' \arrow[l] \end{tikzcd}$$
is snc degeneration

$\Rightarrow \quad (X_0, \mathcal{M}_{X_0}) \longrightarrow \mathrm{Spec}(\mathbb{N} \to \mathbb{C})$ log smooth

---

Explicit meaning of log-smoothness here:

<u>Friedman 1983</u>: $X_0 \subseteq X$ nc degeneration $\Rightarrow X_0$ is <u>d-semistable</u>

"d-semistable" means: $\underbrace{\mathcal{E}xt^1_{\mathcal{O}_{X_0}}(\Omega^1_{X_0}, \mathcal{O}_{X_0})}_{\substack{\text{is an invertible sheaf on the} \\ \text{sing. locus of any nc variety}}} \in \mathrm{Pic}(\underset{\substack{\setminus \\ g=3 \text{ curve!}}}{(X_0)_{sing}})$ is trivial

<u>Kawamata/Namikawa 1994</u>: $X_0$ nc var. is d-semistable

$\Leftrightarrow \exists$ log-str. $\mathcal{M}_{X_0}$ on $X_0$ <u>and</u> a log smooth morph. $(X_0, \mathcal{M}_{X_0}) \to \mathrm{Spec}(\mathbb{N} \to \mathbb{C})$

s.th. locally $\mathbb{N}^r = \overline{\mathcal{M}}_{X_0,x} \xleftarrow{\text{diag}} \mathbb{N}$

<u>Application</u>: (Friedman's Thm) $X_0$ d-semistable K3 $(\omega_X = \mathcal{O}_X, b_1 = 0)$

$\Rightarrow X_0$ is smoothable / fits into nc degeneration

Rem: In higher dims more subtle. (cf. recent work of Felten/Filip/Ruddat)

**Note:** morphism $(X_0, \mathcal{M}_{X_0}) \xrightarrow{f} \operatorname{Spec}(\mathbb{N} \to \mathbb{C}) = (\operatorname{Spec} \mathbb{C}, \mathbb{N} \oplus \mathbb{C}^*)$
is uniquely given by $\rho \in \Gamma(X_0, \mathcal{M}_{X_0})$. $[\rho = f^b((1,1) \in \mathbb{N} \oplus \mathbb{C}^*)$