3) The flow tree formula for quiver DT invariants.

Joint with H. Argüz
2102.11200

(Q, W) quiver with potential $\quad Q_0 = \{\text{Vertices of } Q\}$

$N := \mathbb{Z}^{Q_0} \ (= \Gamma) \quad Q_1 = \{\text{Edges of } Q\}$

$\gamma \in N$

$$\mathcal{M}_\gamma^Q = \frac{\prod_{(\alpha : i \to j) \in Q_1} \mathrm{Hom}(\mathbb{C}^{\gamma_i}, \mathbb{C}^{\gamma_j})}{\prod_{i \in Q_0} GL(\mathbb{C}^{\gamma_i})}$$

Moduli stack of quiver representations of dimension vector γ

$N = \mathbb{Z}^{Q_0}$

$M_{\mathbb{R}} := \mathrm{Hom}(N, \mathbb{R}) \simeq \mathbb{R}^{Q_0}$

$$M_{\mathbb{R}} \longrightarrow \mathrm{Stab}(\mathcal{E}_{(Q,W)})$$

$$\theta \longmapsto \Big(\mathcal{A} = J_{(Q,W)}\text{-mod}\,,\ Z \in \mathrm{Hom}(N, \mathbb{C}) \Big)$$

$$\gamma \mapsto -\theta(\gamma) + i\,|\gamma|$$

$$\shortparallel$$

$$-\sum_j \theta_j \gamma_j + i \sum_j \gamma_j$$

$N = \bigoplus_{j \in Q_0} \mathbb{Z} e_j$

$$Z(e_j) = -\theta(e_j) + i$$

From $\underline{\text{now on}}$: given $\gamma \in N$, only consider

$$\theta \in \gamma^{\perp} = \{ \theta \in M_{\mathbb{R}} \mid \theta(\gamma) = 0 \}$$

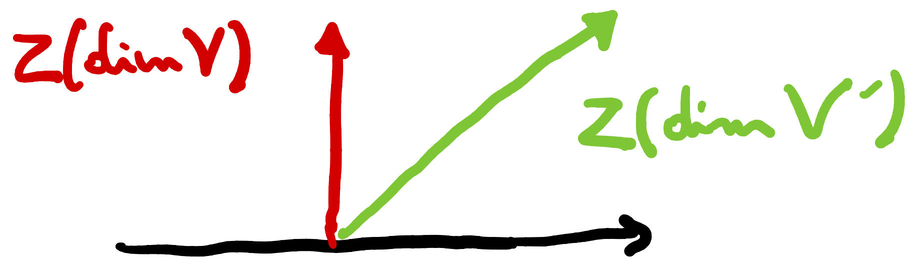
$\uparrow$ Hyperplane where $\mathrm{Re}(Z(\gamma)) = 0$

$Z(\gamma) \in i\,\mathbb{R}$

$\theta \in \gamma^{\perp}$

V representation of dim γ

V θ- semistable (stable) $\iff \forall V' \subsetneq V,\ \theta(\dim V') \underset{(<)}{\leq} 0$



$$\mathcal{M}^{\theta\text{-sst}}_{\gamma,Q} = \Big(\prod_{(\alpha: i\to j)\in Q_1} \mathrm{Hom}(\mathbb{C}^{\gamma_i}, \mathbb{C}^{\gamma_j})\Big) /\!\!/_{\theta} \prod_{i\in Q_0} GL(\mathbb{C}^{\gamma_i})$$

↳ quasiprojective variety over $\mathbb{C}$

Trace function $\mathrm{Tr}\,W : \mathcal{M}^{\theta\text{-sst}}_{\gamma,Q} \longrightarrow \mathbb{C}$

$$\mathcal{M}^{\theta\text{-sst}}_{\gamma,(Q,W)} = \mathrm{Crit}(\mathrm{Tr}\,W)$$

Quiver DT invariants

$$\Omega_\gamma^\theta = \begin{cases} 0 & \text{if } \mathcal{M}_\gamma^{\theta\text{-st}} = \emptyset \\ \chi\left(\mathcal{M}_\gamma^{\theta\text{-sst}}, \varphi_{\mathrm{Tr}(W)}\left(IC_{\mathcal{M}_\gamma^{\theta\text{-sst}}}\right)\right) \in \mathbb{Z} & \end{cases}$$

<u>Example:</u> $W=0$ $\gamma \in N$ primitive : $\mathcal{M}_\gamma^{\theta\text{-sst}} = \mathcal{M}_\gamma^{\theta\text{-st}}$

$$\Omega_\gamma^\theta = (-1)^{\dim \mathcal{M}_\gamma^{\theta\text{-st}}} \chi\left(\mathcal{M}_\gamma^{\theta\text{-st}}\right) \in \mathbb{Z}$$

Rational DT invariant

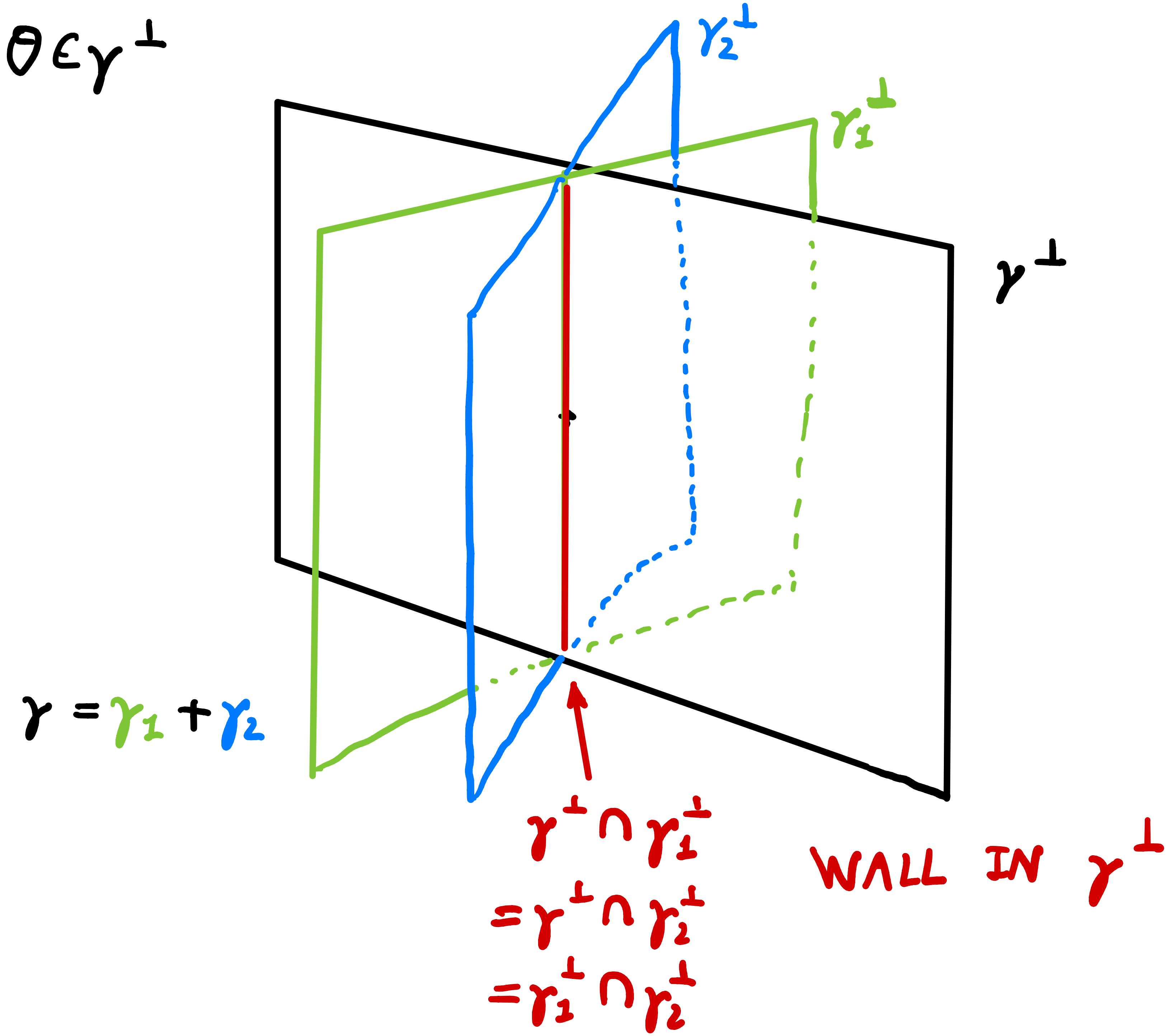
$$\overline{\Omega}_\gamma^\theta = \sum_{\gamma = k\gamma'} \frac{1}{k^2} \Omega_{\gamma'}^\theta$$

$$\left[\quad \Omega_\gamma^\theta = \sum_{\gamma = k\gamma'} \frac{\mu(k)}{k^2} \overline{\Omega}_{\gamma'}^\theta \quad \right]$$

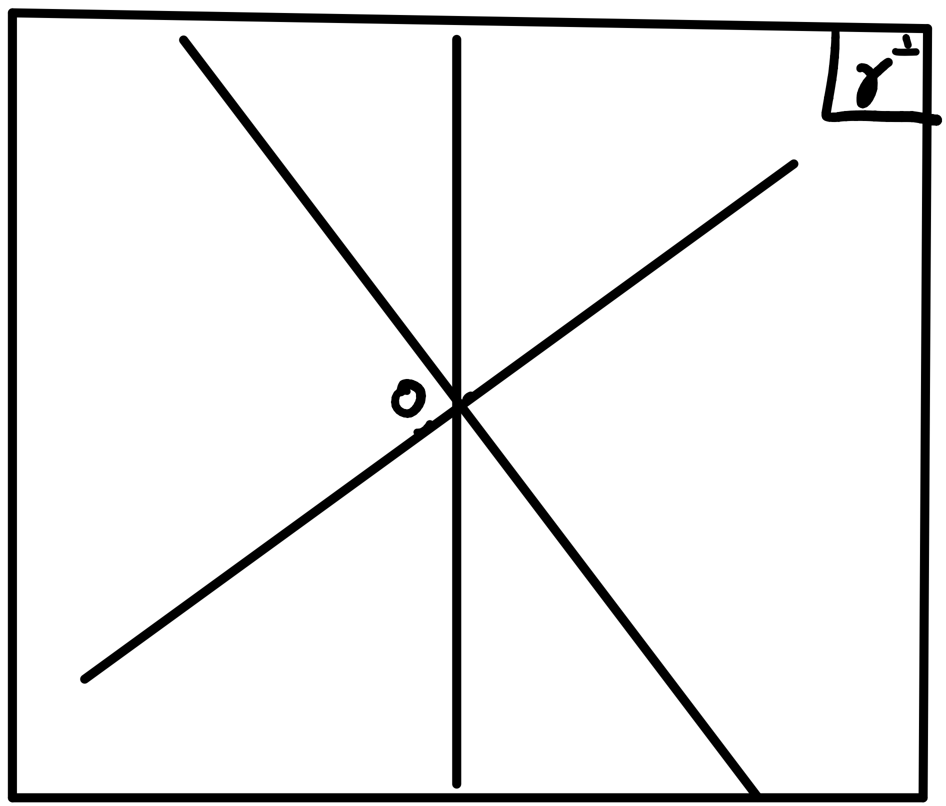
Möbius inversion

$\mu(p_1 \cdots p_r) = (-1)^r$

$\mu = 0$ else


$\theta \in \gamma^\perp$
$\gamma_2^\perp$
$\gamma_1^\perp$
$\gamma^\perp$
$\gamma = \gamma_1 + \gamma_2$
$\gamma^\perp \cap \gamma_1^\perp$
$= \gamma^\perp \cap \gamma_2^\perp$
$= \gamma_1^\perp \cap \gamma_2^\perp$
WALL IN $\gamma^\perp$

WALL AND CHAMBER STRUCTURE



ATTRACTOR DT INVARIANTS

CY3 Euler form $\omega : N \times N \to \mathbb{Z}$

$$\omega(e_i, e_j) = a_{ij} - a_{ji} \qquad a_{ij} = \#\ \text{arrows}_{i \to j} \text{ in } Q$$

<u>Definition</u>

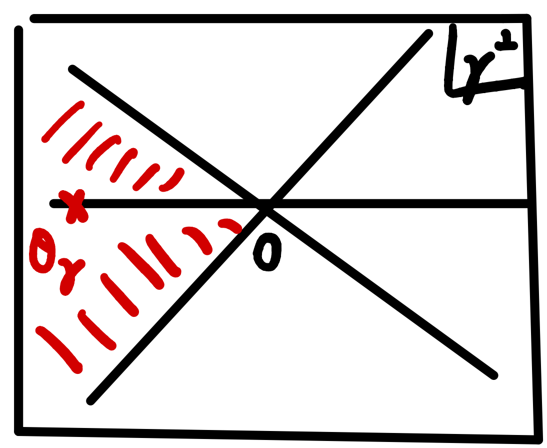
1) Attractor point:

$$\theta_\gamma := \omega(\gamma, -) = \iota_\gamma \omega \in \gamma^\perp \subset M_{\mathbb{R}}$$

because $\omega(\gamma, \gamma) = 0$

2) Attractor DT invariant

$$\Omega_\gamma^* := \Omega_\gamma^{\theta_\gamma}$$



TRIVIAL ATTRACTOR DT INVARIANTS

<u>Definition</u>: (Q,W) has trivial attractor DT invariants if

$$\Omega^{*}_{e_i} = 1 \quad \forall i \in Q_0 \qquad \Omega^{*}_{\gamma} = 0 \quad \forall \gamma \in N$$

s.t. $\gamma \neq e_i$

$\gamma \notin \operatorname{Ker} \omega$

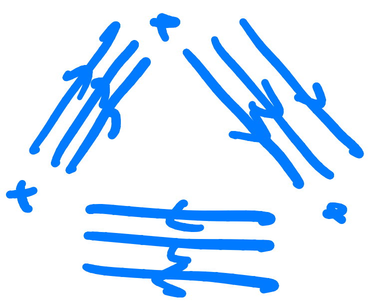
<u>Theorem</u>: (Q,W) has trivial attractor DT invts if Q is acyclic (Bridgeland) or more generally if (Q,W) admits a reddening sequence (Lang Mou)

<u>Conjecture</u> [Beaujard-Manschot-Pioline, Mozgovoy-Pioline]

X toric CY 3-fold $\quad (Q,W)$ has trivial attractor DT invts

$\hookrightarrow (Q,W)$

<u>True</u> for $X = K_{\mathbb{P}^2}$ [B-Descombes-Le Floch-Pioline]

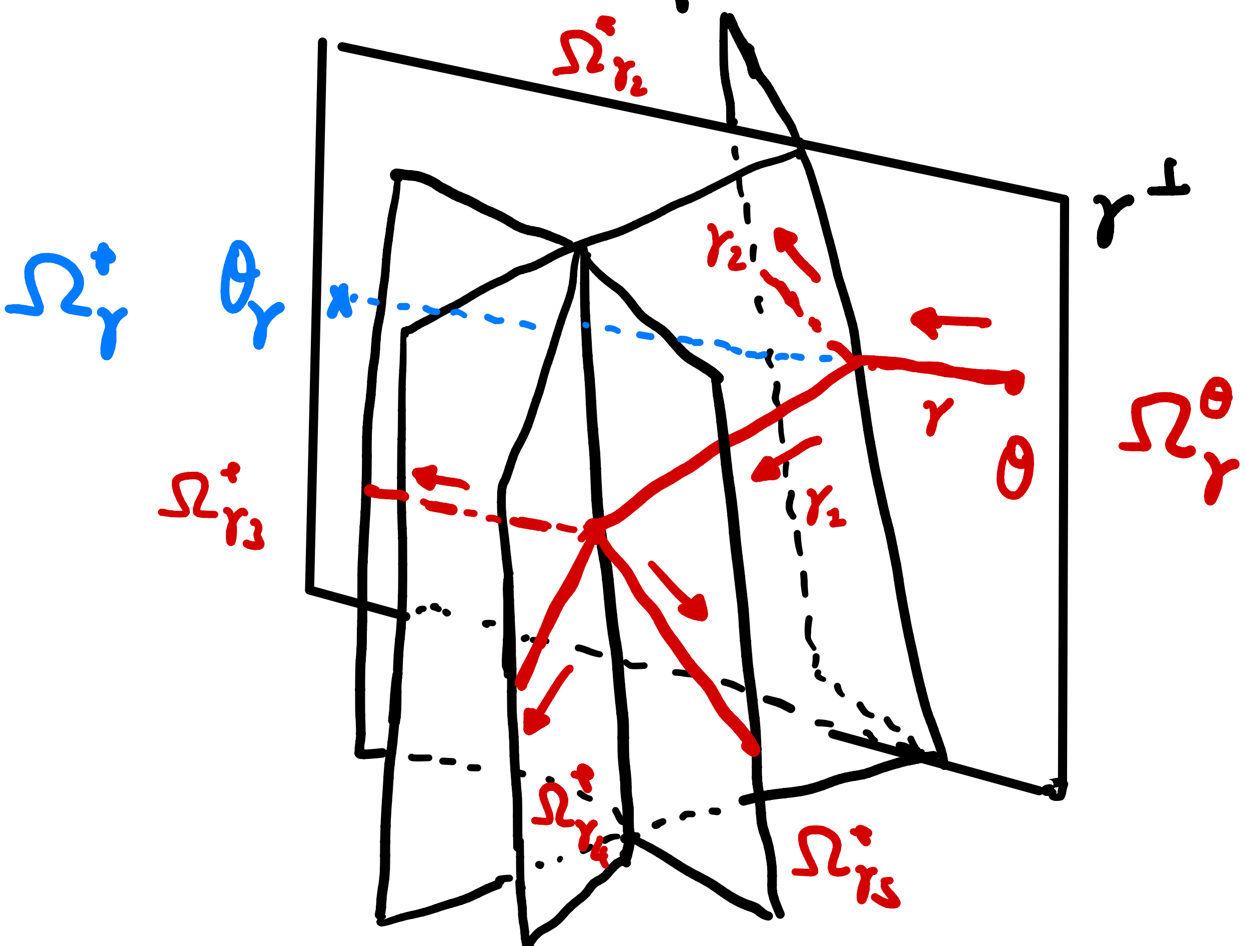
$Q =$ 

$$W = \sum_{i,j,k} \varepsilon_{ijk}\, x_i y_j z_k$$

Iterated application of the wall-crossing formula [Kontsevich-Soibelman 2013]

$$\bar{\Omega}^{\theta}_{\gamma} = \sum_{\gamma=\gamma_1+\dots+\gamma_r} \frac{1}{|\mathrm{Aut}\{\gamma_i\}_i|} F^{\theta}_r(\gamma_1,\dots,\gamma_r) \prod_{i=1}^{r} \bar{\Omega}^{*}_{\gamma_i}$$

$$F^{\theta}_r(\gamma_1,\dots,\gamma_r) = \sum_{T} F^{\theta}_{r,T}(\gamma_1,\dots,\gamma_r)$$



$\gamma = \gamma_1 + \gamma_2$

$\gamma_1 = \gamma_3 + \gamma_4 + \gamma_5$

Tropical Curves

Ω^{θ}_{γ}

$L_\gamma W$

GOAL: FLOW TREE FORMULA

= EXPICIT FORMULA FOR

$F_r^{\theta}(\gamma_1, \dots, \gamma_r)$

CONJECTURED BY ALEXANDROV-PIOLINE

IN THE PHYSICS LITERATURE

[TOY MODEL OF ATTRACTOR MECHANISM FOR BLACK HOLES IN 4d $\mathcal{N}=2$ SUPERGRAVITY]

Fix $\gamma_1, \dots, \gamma_r \in N$ $\quad \theta \in \gamma^{\perp}$

$\mathcal{N} := \bigoplus_{i=1}^{r} \mathbb{Z} f_i \xrightarrow{f} N$ $\qquad$ dual $\quad M_{\mathbb{R}} \xrightarrow{\alpha} \mathcal{M}_{\mathbb{R}}$

$f_i \longmapsto \gamma_i$ $\qquad M_{\mathbb{R}} = \mathrm{Hom}(N, \mathbb{R})$, $\quad \mathcal{M}_{\mathbb{R}} = \mathrm{Hom}(\mathcal{N}, \mathbb{R})$

$\theta \longmapsto \alpha(\theta)$

$\tilde{\omega} = p^*\omega : \mathcal{N} \times \mathcal{N} \to \mathbb{Z}$

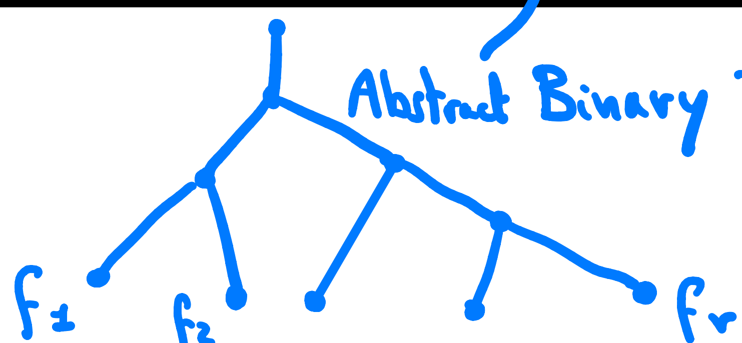
skew-symmetric form

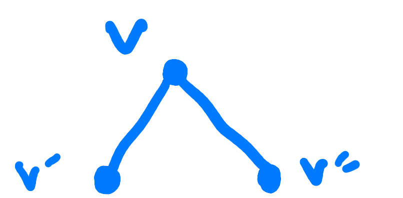
<u>Thm</u> [Argüz-B] (2021) Pick $\tilde{\theta} \in \left(\sum_i f_i\right)^{\perp}$ small generic perturbation of $\alpha(\theta)$

Then

$$F_r^{\theta}(\gamma_1, \dots, \gamma_r) = \sum_{T \in \mathcal{T}_r} \prod_{v \in V_T^{\circ}} \varepsilon_{T,v}^{\tilde{\theta}} (-1)^{\tilde{\omega}(f_{v'}, f_{v''})} \tilde{\omega}(f_{v'}, f_{v''})$$

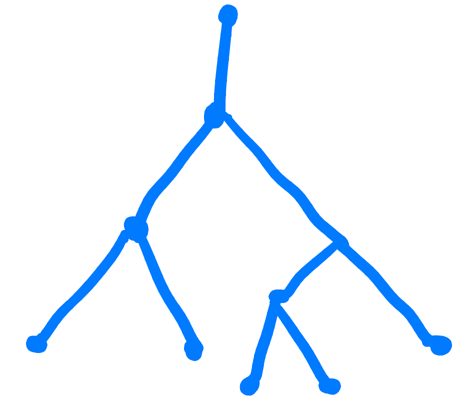
Abstract Binary Trees



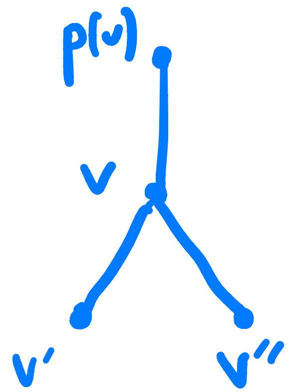


$f_v = f_{v'} + f_{v''}$

$\varepsilon^{\tilde\Theta}_{T,v} \in [0, 1, -1]$?

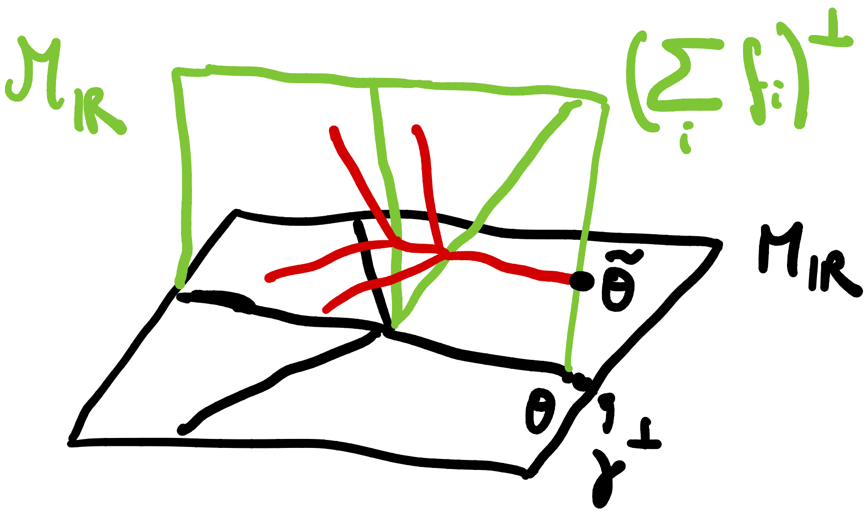


Discrete attractor flow v root $\tilde\Theta_v := \tilde\Theta$



$$\tilde\Theta_v := \tilde\Theta_{p(v)} - \frac{\tilde\Theta_{p(v)}(f_{v'})}{\tilde\omega(f_{v}, f_{v'})}\,\iota_{f_v}\tilde\omega$$

$$\varepsilon^{\tilde\Theta}_{T,v} := -\frac{1}{2}\left[\mathrm{sign}\big(\underbrace{\tilde\Theta_{p(v)}(f_{v'})}_{\neq 0}\big) + \mathrm{sign}\big(\tilde\omega(f_{v'}, f_{v''})\big)\right] \in \{0, 1, -1\}$$



Sketch of the proof:

General Trees perturbed to binary Trees

Signs $\varepsilon \leftrightarrow$ realizability of T as a tropical curve

NEXT TIME:

FROM TROPICAL CURVES TO $\mathbb{C}$ CURVES [LOG GW]