1. Tropicalization

Traditionally: $K = \mathbb{C}\{\{t\}\} = \bigcup_n \mathbb{C}((t^{1/n}))$ Puiseux series $\left(= \text{alg. closure of } \mathbb{C}((t))\right)$,
or other alg. closed valued $\mathbb{C}$-algebra

$I \subseteq K[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ ideal, $X = V(I) \subseteq (K^\times)^n \xrightarrow{\text{val}} \mathbb{R}^n$

$$\boxed{\begin{aligned} \operatorname{Trop}(X) &= \overline{\operatorname{val}(X)} \\ &= \{w \in \mathbb{R}^n \mid \operatorname{in}_w(I) \neq (1)\} \\ &= \text{"} V(\operatorname{trop}(I)) \text{"} \end{aligned}}$$

$\operatorname{in}_w I$: initial ideal for weight w

$\operatorname{trop}(I) \subseteq (\mathbb{R}, \max, +)[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$

Convergent case: $X = \{X_t\}_{|t| < \varepsilon}$, $X_t \subseteq (\mathbb{C}^*)^n$

Then $\boxed{\operatorname{Trop} X = \lim_{t \to 0} \operatorname{Log}_t(X_t)}$ $\quad \operatorname{Log}_t(z_1, \dots, z_n) = (\log_t |z_1|, \dots, \log_t |z_n|)$

In log geometry: $X = (\underline{X}, \mathcal{M}_X)$ fine log space

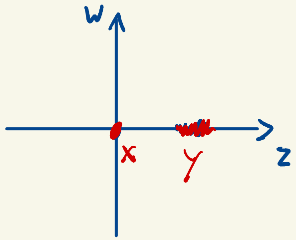
$\forall x$: $P_x = \overline{\mathcal{M}}_{X,x}$ toric monoid, $P_x^\times = \{0\}$

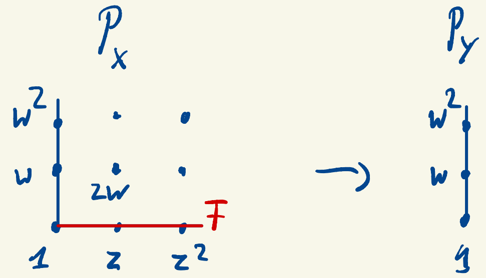
schematically: $x \in \operatorname{cl}(y) \Rightarrow P_x = \overline{\mathcal{M}}_{X,x} \xrightarrow[\chi]{\text{generize}} \overline{\mathcal{M}}_{X,y} = P_y$

What is χ? $F = \chi^{-1}(0) \subseteq P_x$ face, χ induces iso $(P_x + F^{gp})/F^{gp} \to P_y$

i.e. χ = projection along face F (of "exponents" becoming invertible at y)

Expl: $X = \mathbb{A}^2$





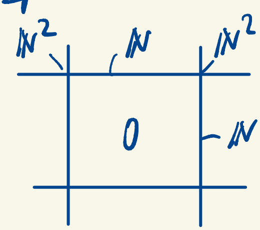
__Dually:__ $\sigma_x = \mathrm{Hom}(P_x, \mathbb{R}_{\geq 0})$ $\quad$ $(P = \sigma_{\mathcal{X}}^{\vee})$

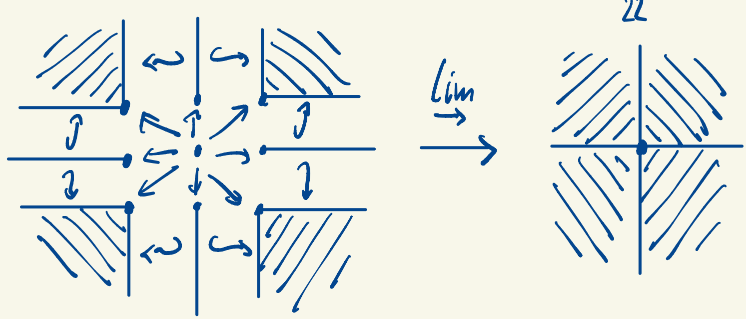
then $\chi^t : \sigma_y = \sigma_x \cap F^{\perp} \hookrightarrow \sigma_x$ $\quad$ face inclusion

__Tropicalization of $X = (\underline{X}, \mathcal{M}_X)$:__ $\quad$ $\boxed{\Sigma_X = \mathrm{Trop}(X) := \varinjlim \sigma_x}$

is a (generalized) cone complex (of rat'l polyh. cones)

Expl: $X = \mathbb{P}^1 \times \mathbb{P}^1$ $\quad$ (σ_x) $\quad$ fan of $\mathbb{P}^1 \times \mathbb{P}^1$ $\cong$



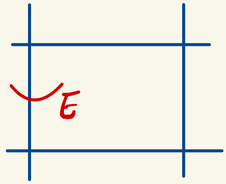


Prop: $X = TV(\Sigma)$ toric var. $\Rightarrow$ $\Sigma_X = \Sigma$, but forget the embedding of Σ into $N_{\mathbb{R}} \simeq \mathbb{R}^n$

fan

Note also: $X = Bl_1\, \mathbb{P}^1 \times \mathbb{P}^1$

$D = \bigcup_4 \mathbb{P}^1$



$\Rightarrow \Sigma_X \simeq \Sigma_{\mathbb{P}^1 \times \mathbb{P}^1}$

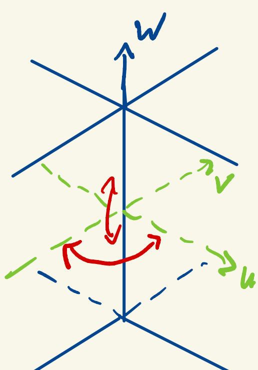
This is actually good: Σ_X only captures the combinatorics of the situation, not the finer analytic or algebraic geometry, hence has a more fundamental and universal character: $\Sigma_{\mathbb{P}^1 \times \mathbb{P}^1}$ always appear in an nc situation with 4 divisors $D_1, \ldots, D_4$ meeting circularly: $D_i \cap D_j \neq \emptyset \iff |i-j| \leq 1$ or $\{i,j\} = \{1,4\}$.

Cones can <u>self-intersect</u> if $\bar{\mathcal{M}}_X$ is only <u>locally constant</u> rather than constant along a <u>stratum</u> of X.

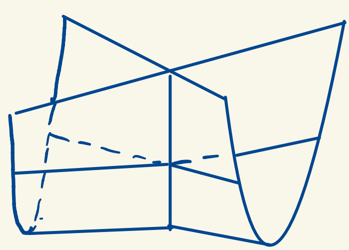
[a maximal locally closed subset $Z \subset X$ with $\bar{\mathcal{M}}_X|_Z$ locally constant]

Expl: Whitney umbrella $\bar{D} = V(zx^2 - y^2) \subseteq \mathbb{C}^3$

$$\begin{array}{ccc} \bar{D} & & \mathbb{C}^3 \\ \cup\!\! \mid & & \cup\!\! \mid \\ D = \bar{D} \setminus V(z) & \subseteq & \mathbb{C}^*_z \times \mathbb{C}^2_{xy} = X \end{array}$$



$\xrightarrow{\quad /\mathbb{Z}/2 \quad}$



$$D \simeq (\mathbb{C}^*_w \times V(uv))/\mathbb{Z}/2 : \quad \mathbb{C}[w^{\pm}, u, v]/(uv) \leftarrow \mathbb{C}[x,y,z]/(zx^2-y^2)$$

$\boxed{\text{II.4}}$

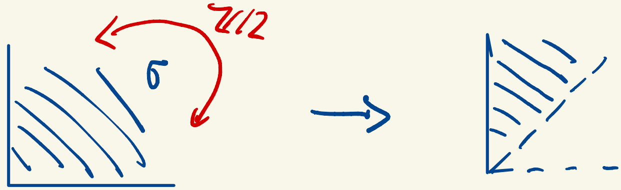
$\mathbb{Z}/2$-action: $(u,v,w) \leftrightarrow (v,u,-w)$

invts: $uv, w^2, u+v, (u-v)\cdot w$

$$u+v \longleftarrow\!\shortmid\ x$$
$$w\cdot(u-v) \longleftarrow\!\shortmid\ y$$
$$w^2 \longleftarrow\!\shortmid\ z$$

Tropicalize X

$\mathcal{M}_X = \mathcal{M}_{(X,D)}$



$\Rightarrow \quad \mathrm{Trop}(X) = \mathbb{R}^2_{\geq 0}/(\mathbb{Z}/2)$

Shows: $\mathrm{Trop}(X)$ generally is only a generalized cone complex, i.e. a topological space $|\Sigma_X|$ locally given as a colimit (direct limit) of a diagram of rational polyhedral cones with arrows face inclusions, up to equivalence of diagrams.

2. Log smooth curves

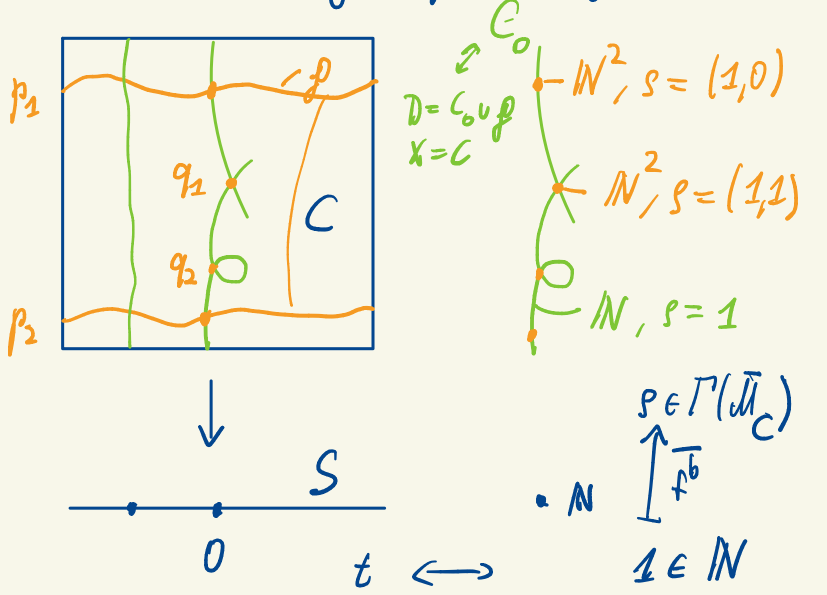
... serve as domains for stable log maps in log GW theory.

nc degeneration of smooth marked curves

$\Downarrow$

log morphism

$C \longrightarrow S$



local models:

- at a general or generic pt η: $\quad (t,z) \mapsto t \quad$ $\bar{\mathcal{M}}_{C_0,\eta} = \bar{\mathcal{M}}_{C,\eta} = \langle t \rangle \simeq \mathbb{N}$, $\quad t \leftrightarrow 1$

- at a marked point $p = (z=0)$, $\quad (t,z) \mapsto t \quad$ $\bar{\mathcal{M}}_{C_0,p} = \bar{\mathcal{M}}_{C,p} = \langle z,t \rangle \simeq \mathbb{N}^2$, $\quad t \leftrightarrow (1,0)$

- at a nodal point q: $\quad (z,w) \mapsto zw \quad$ $\bar{\mathcal{M}}_{C_0,q} = \bar{\mathcal{M}}_{C,q} = \langle z,w \rangle \simeq \mathbb{N}^2$, $\quad t \leftrightarrow (1,1)$

Generally: $C \xrightarrow{\pi} \mathrm{Spec}(Q \to \mathbb{C})$ log smooth, fs,
C reduced, π integral [$\mathbb{C}[Q] \to \mathbb{C}[P]$ flat in local toric description]

$\Rightarrow$ C is a nodal curve.

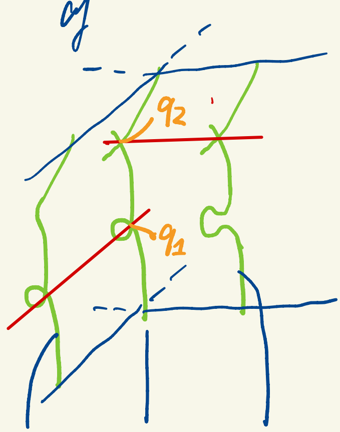
Moreover, locally on C:

- at η: $\bar{\mathcal{M}}_{C,\eta} = Q$
- at p: $\bar{\mathcal{M}}_{C,p} = Q \oplus \mathbb{N}$ [vanishing order at p]
- at q: $\bar{\mathcal{M}}_{C,q} = Q \oplus_{\mathbb{N}} \mathbb{N}^2 = Q \oplus \mathbb{N}^2 / ((s,0) \sim (0,(1,1)))$ [for some $s \in Q$]

$s \leftarrow 1 \mapsto (1,1)$

Universal case: $Q = \mathbb{N}^{\#\text{nodes of } C}$

Interpretation: This is the log structure induced from the total space of the (semi-)universal deformation of C, with divisor the union of singular curves.



Log structure: $\bar{\mathcal{M}}_{C_0,\eta} = \mathbb{N}^r$

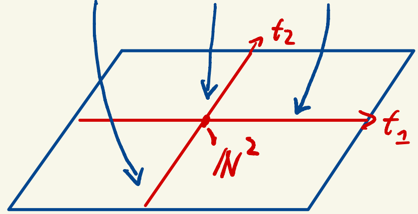
$r = \#$ nodes $\quad \bar{\mathcal{M}}_{C_0,p} = \mathbb{N}^r \oplus \mathbb{N}$

at i-th node: $\bar{\mathcal{M}}_{C_0,q_i} = \mathbb{N}^r \oplus_{\mathbb{N}} \mathbb{N}^2$

$e_i \leftarrow 1 \mapsto (1,1)$

$t_1 \neq 0$: first node smoothes out

$t_2 \neq 0$: second " " "



<u>Thus</u>: i-th coordinate of $Q = \mathbb{N}^r$ captures the smoothing parameter of the i-th node.

<u>Stable cases</u>: Deligne-Mumford moduli space (stack) $\mathcal{M}_{g,k}$

$$\begin{tikzcd} \mathcal{C} \arrow[d] & \supseteq & \tilde{D} \arrow[d] \\ \mathcal{M}_{g,k} & \supseteq & D \end{tikzcd}$$

nc divisor of singular curves

$\rightsquigarrow$ log smooth morphism $(\mathcal{C}, \mathcal{M}_{(\mathcal{C},\tilde{D})}) \to (\mathcal{M}_{g,k}, \mathcal{M}_{(\mathcal{M}_{g,k}, D)})$

<u>Universal property</u>:

$\forall \pi$ $\exists$ unique diagram:

$$\begin{tikzcd} (C_0, \mathcal{M}_{C_0}) \arrow[r] \arrow[d, "\pi"'] & (C_0, \mathcal{M}^{univ}_{C_0}) \arrow[d] \\ \mathrm{Spec}(Q \to \mathbb{C}) \arrow[r] & \mathrm{Spec}(\mathbb{N}^r \to \mathbb{C}) \end{tikzcd}$$

$$Q \longleftarrow \mathbb{N}^r$$
$$g_i \longleftarrow\!\shortmid e_i$$

3. Tropicalization of log smooth curves

Over standard log point

cone complex — polyhedral complex

$$\begin{array}{ccc} C = (\underline{C}, \mathcal{M}_C) & & \Sigma_C = \mathrm{Cone}(\Gamma) \supseteq \Gamma \\ \downarrow \pi_C & \overset{\text{trop}}{\rightsquigarrow} & \downarrow \pi^{\text{trop}} \qquad \downarrow \\ \mathrm{Spec}(\mathbb{N} \to \mathbb{C}) & & \tau = \mathbb{R}_{\geq 0} \ni 1 \end{array}$$

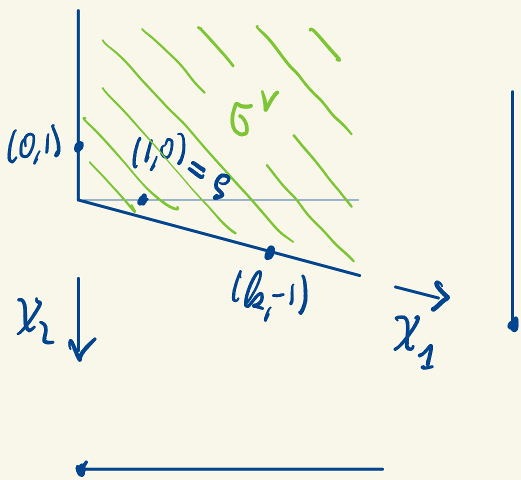
At node q: $\overline{\mathcal{M}}_{C,q} \simeq \mathbb{N} \oplus_{\mathbb{N}} \mathbb{N}^2 \overset{\phi}{\simeq} \langle (1,0), (0,1), (k,-1) \rangle = \sigma^\vee_{\mathbb{Z}}$

2d cone $\sigma \in \Sigma_C$

$k \leftarrow 1 \mapsto (1,1)$

$$\begin{aligned} \phi: (1,0) &\longmapsto (1,(0,0)) \\ (0,1) &\longmapsto (0,(0,1)) \\ (k,-1) &\longmapsto (0,(1,0)) \end{aligned}$$

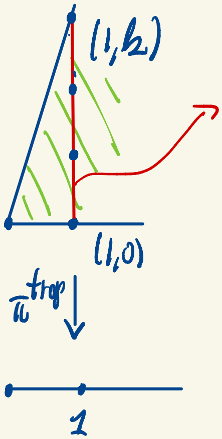
$(a+bk, c-b) \longleftarrow (a,(b,c))$



χ_i: projections along faces = generization maps $\overline{\mathcal{M}}_{C,q} \longrightarrow \overline{\mathcal{M}}_{C,\eta_i}$

Dualize to compute $\Sigma_C \longrightarrow \tau = \mathbb{R}_{\geq 0}$:

$|\Sigma_C| \supseteq \sigma =$

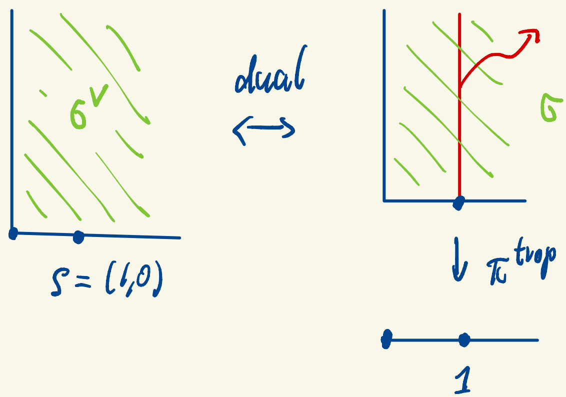


interval $[1,k] \subseteq \Gamma = (\pi^{trop})^{-1}(1)$

vertices $1, k \longleftrightarrow$ branches of C at q

[π^{trop} is dual to $\mathbb{R}_{\geq 0} \longrightarrow \sigma^\vee$
$1 \longmapsto \rho = (1,0)$]

<u>At marked point p:</u>



ray $\mathbb{R}_{\geq 0} \subseteq |\Sigma_C|$

Thus, Γ = dual graph of C

$\Gamma \cong \pi_{trop}^{-1}(1)$

vertices $\longleftrightarrow$ irred. cmps. of C

bounded edges $\longleftrightarrow$ nodes

unbounded edges $\longleftrightarrow$ marked points

Edges have a $\mathbb{Z}$-affine structure $\rightsquigarrow$ metric graph

(i.e. an integral length)

Slogan: The tropicalization of a log smooth curve over a standard log point is a metric graph with bounded edges of integral length k corresponding to nodes with local model $xy = t^k$, and unbounded edges corresponding to the other logarithmically special points ("marked points").

Over $\mathrm{Spec}(Q \to \mathbb{C})$:

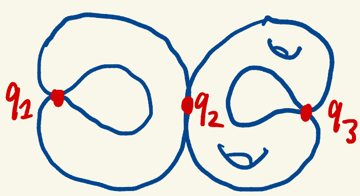
$$\begin{array}{rl} & \Sigma_C \\ & \downarrow \\ \text{cone} & \tau = \mathrm{Hom}_{\mathrm{mon}}(Q, \mathbb{R}_{\geq 0}) \end{array}$$

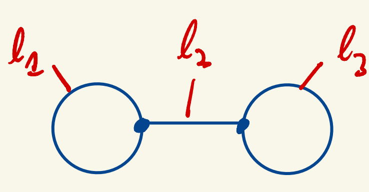
is a family of metric graphs parametrized by τ with edge length varying with integral derivatives

Note: Faces of $\tau \overset{1:1}{\longleftrightarrow}$ collapse of some edges

Universal case: $\tau = \mathbb{R}_{\geq 0}^{\#\text{nodes}}$ tropical moduli space for the given Γ

Expl:

$C =$ 

$\xrightarrow{\text{trop}}$ 

$\tau = \{(l_1, l_2, l_3)\} = \mathbb{R}^3_{\geq 0}$