Let Y be a conical symplectic resolution.

There exists a quantum multiplication $*_q$ on $H^\bullet_{T\times\mathbb{C}^\times}(Y,\mathbb{C})$ depending on $q\in H^2(Y,\mathbb{C}^\times)^{reg}$.

Problems

① Describe this quantum multiplication using geometric representation theory

② Study the compactified parameter space for these algebras.

Plan

① Generalities on symplectic resolutions and quantum cohomology

② Quantum multiplications for quiver varieties and affine Grassmannian slices.

③ Compactified parameter spaces for quantum cohomology algebras.

Symplectic resolutions

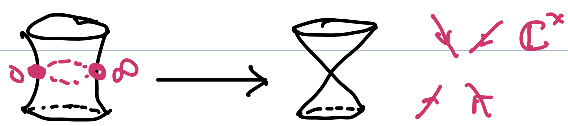
A <u>symplectic resolution</u> is $\pi : Y \to X$ varieties/$\mathbb{C}$
s.t. (i) Y smooth, symplectic
(ii) X affine, normal, Poisson
(iii) π projective, birational, Poisson

Y is called <u>conical</u> if we are given $\mathbb{C}^\times \circlearrowright Y, X$
s.t. · $\mathbb{C}[X] = \bigoplus_{n \geq 0} \mathbb{C}[X]_n$ $\quad \mathbb{C}[X]_0 = \mathbb{C}$
$\mathbb{C}^\times$ contracts X to a point $0 \in X$
· $\mathbb{C}^\times$ scales the symplectic form

Y is called <u>Hamiltonian</u> if we are given
$T \circlearrowright Y, X$ with moment map $X \to \mathfrak{t}^*$ s.t. Y^T finite

Eg $\qquad T^*\mathbb{P}^1 \longrightarrow \mathcal{N}$

$\left\{\left[\begin{smallmatrix} z_1 & \\ & z_2 \end{smallmatrix}\right]\right\} = (\mathbb{C}^\times)^2/\mathbb{C}^\times = T \circlearrowright$



$\{(A,L) \in M_2(\mathbb{C}) \times \mathbb{P}^1 : AL = 0, A\mathbb{C}^2 \subseteq L\} \to \left\{\left[\begin{smallmatrix} w & u \\ v & -w \end{smallmatrix}\right] : w^2 + uv = 0\right\}$

Generalizations

(i) $\quad T^*G(k,n) \longrightarrow \{A \in M_n : \mathrm{rk}\, A \leq k,\ A^2 = 0\}$ $\quad k \leq \frac{n}{2}$

$T^*Fl_n \to \mathcal{N}$ $\quad T^*(\text{partial flag variety})$

G reductive $\quad T^*G/B \to \mathcal{N}_{\mathfrak{g}}$ $\quad T^*G/P$

(ii) hypertoric varieties $\quad \dim Y = 2\dim T$

(iii) Quiver varieties

Let (I, E) be a directed graph, usually ADE Dynkin

$\underline{v}, \underline{w} \in \mathbb{N}^I$ dimension vectors

$\bullet \to \bullet \to \bullet$
$$\begin{tikzcd} V_1 \arrow[r] \arrow[d] & V_2 \arrow[r] \arrow[d] & V_3 \arrow[d] \\ \boxed{W_1} & \boxed{W_2} & \boxed{W_3} \end{tikzcd}$$
$V_i = \mathbb{C}^{v_i}$

$W_i = \mathbb{C}^{w_i}$

$$N = \bigoplus_{e\in E} \mathrm{Hom}(V_{t(e)}, V_{h(e)}) \oplus \bigoplus_{i\in I} \mathrm{Hom}(V_i, W_i)$$

$G = \prod_{i\in I} GL(V_i) \circlearrowright N$

$\mathrm{Hom}(V_1, V_2)^* = \mathrm{Hom}(V_2, V_1)$

$T^*N = N \oplus N^*$ double arrows $\quad \Phi: T^*N \to \mathfrak{g}^* = \mathfrak{g}$

$\chi: G \to \mathbb{C}^\times \quad \chi((g_i)) = \prod \det g_i$

$$Y = \Phi^{-1}(0) /\!\!/_{\chi} G \longrightarrow X = \Phi^{-1}(0) /\!\!/ G$$

$T = T_W = (\mathbb{C}^\times)^n \quad n = \sum w_i \quad \mathbb{C}^\times$ scales N

- not always birational
- T action doesn't always have finitely many fixed pts

eg

① $\bullet$
$$\textcircled{1} \to \boxed{n}$$
$N = \mathbb{C}^n \quad G = \mathbb{C}^\times \quad Y = T^*\mathbb{P}^{n-1}$

② $\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet$
$$\textcircled{1} \to \textcircled{2} \to \cdots \to \textcircled{n-1} \to \boxed{n}$$
$Y = T^*Fl_n$

(3) $\circlearrowleft \boxed{n}$ ⟶ $\boxed{1}$ $\quad N = M_n\mathbb{C} \oplus \mathbb{C}^n \qquad Y = \mathrm{Hilb}^n \mathbb{C}^2$

$\circlearrowleft$

The data (G,N) above gives a quiver gauge theory which is a 3d $N=4$ QFT

$T^*N /\!/\!/\!/_{(0,\chi)} G = \Phi^{-1}(0) /\!/_{\chi} G$ $\quad$ Higgs branch

(G,N)

any reductive group $\qquad$ any rep.

BFN construction $\quad$ Coulomb branch $\qquad$ symplectic duality

$G \circlearrowright N \quad \Phi: N \oplus N^* \to \mathfrak{g}^* \quad \Phi(\alpha, v)(X) = \alpha(Xv)$

(iv) Affine Grassmannian slices

G reductive group $\qquad G = GL_m$

$Gr = \underset{\mathrm{Maps}(\mathbb{C}^\times, G)}{G(\mathbb{C}[t,t^{-1}])} \big/ \underset{\mathrm{Maps}(\mathbb{C}, G)}{G(\mathbb{C}[t])} = \{ (P, \varphi) : P \text{ principal } G\text{-bundle on } \mathbb{P}^1 \}$

$= G[\mathbb{C}((t))]/G(\mathbb{C}[[t]]) \qquad \varphi : P|_{\mathbb{P}^1 \setminus \{0\}} \xrightarrow{\sim} P_0|_{\mathbb{P}^1 \setminus \{0\}}$

$GL_m : \quad Gr = \{ L \subset \mathbb{C}[t,t^{-1}]^m : L \text{ is a } \mathbb{C}[t]\text{-lattice} \}$

For $\mu : \mathbb{C}^\times \to T$ a "coweight of G", we get $t^\mu \in Gr$

λ dominant $\qquad Gr^\lambda = G(\mathbb{C}[t])\, t^\lambda \qquad$ spherical Schubert cell

Eg $G = GL_m \qquad \lambda = (\lambda_1 \geq \dots \geq \lambda_m \geq 0)$

$t^\lambda = \begin{bmatrix} t^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & t^{\lambda_m} \end{bmatrix} \qquad Gr^\lambda = \{ L \subset \mathbb{C}[t]^m : \mathbb{C}[t]^m/L \cong \mathbb{C}[t]/t^{\lambda_1} \oplus \dots \oplus \mathbb{C}[t]/t^{\lambda_m} \}$

$= \{ \mathcal{V} \subset \mathcal{O}_{\mathbb{P}^1}^{\oplus m} : \mathcal{O}_{\mathbb{P}^1}^{\oplus m}/\mathcal{V} \cong \dots \}$

μ dominant $\quad W_\mu = G_1[t^{-1}]\, t^\mu \subset Gr \qquad G_1[t^{-1}] = \ker G(\mathbb{C}[t^{-1}]) \to G$

transverse to $Gr^\lambda \qquad (\mathfrak{g}[[t]], t^{-1}\mathfrak{g}[t^{-1}], \mathfrak{g}((t))) \qquad t^{-1} \mapsto 0$

$W^\lambda_\mu = \overline{Gr^\lambda} \cap W_\mu \quad$ affine Poisson variety

Eg $G = GL_m$ principal G-bundle = rank m vector bundle

$$W_\mu = \{ (\mathcal{V}, \varphi) : \mathcal{V} \cong \mathcal{O}(\mu_1) \oplus \cdots \oplus \mathcal{O}(\mu_m)$$
$$\varphi : \mathcal{V}|_{\mathbb{P}^1 \setminus \{0\}} \xrightarrow{\sim} \mathcal{O}^{\oplus m}_{\mathbb{P}^1 \setminus \{0\}} \text{ + condition about HN flag at } \infty \}$$

$\lambda = \lambda_1 + \dots + \lambda_n$ λ_j dom wt

$(P_m, \varphi_1 \circ \cdots \circ \varphi_m)$ $\overline{Gr^\lambda}$

$$Gr^{\lambda_1} \tilde{\times} \cdots \tilde{\times} Gr^{\lambda_n} = \{ P_0 \overset{\varphi_1}{\dashrightarrow} P_1 \overset{\varphi_2}{\dashrightarrow} P_2 \dashrightarrow \cdots \dashrightarrow P_m : \varphi_j \text{ iso away from } 0 \text{ Hecke type } \lambda_j \}$$

λ is called <u>minuscule</u> if $Gr^\lambda = \overline{Gr^\lambda}$ $\lambda = \omega_k = (1, \dots, 1, 0, \dots, 0)$ $G(k,m)$

$Gr^{\underline{\lambda}} = Gr^{\lambda_1} \tilde{\times} \cdots \tilde{\times} Gr^{\lambda_n} \to \overline{Gr^\lambda}$ is a resolution if λ_j are all minuscule

$W^{\underline{\lambda}}_\mu := Gr^{\underline{\lambda}} \times_{Gr} W_\mu \to W^\lambda_\mu$ is a symplectic resolution

$W_\mu \subset Gr$ $Gr^{\underline{\lambda}} \to \overline{Gr^\lambda} \subset Gr$

Eg GL_m $\lambda = m\omega_1 = (m, 0, \dots, 0)$

$W^{\underline{\lambda}}_0 = T^*Fl_n \to W^\lambda_0 = \mathcal{N}$

$W^{m\omega_1}_\mu \cong$ Slodowy slice $\cap\ \mathcal{N}$ $\mu \vdash m$

$W^\lambda_0 \cong \overline{\mathbb{O}}_\lambda$ nilpotent orbit closure $\lambda \vdash m$

<u>Theorem</u> [Braverman–Finkelberg–Nakajima, N-Weekes]
W^λ_μ is the Coulomb branch of the quiver gauge theory.

G simply-laced semisimple group$/\mathbb{C}$ λ, μ dom. coweights

$\lambda = \sum_{i \in I} w_i \omega_i = \lambda_1 + \cdots + \lambda_n$ $\lambda - \mu = \sum_{i \in I} v_i \alpha_i$

$$\begin{tikzcd} v_1 \arrow[r] \arrow[d] & v_2 \arrow[r] \arrow[d] & v_3 \arrow[d] \\ w_1 & w_2 & w_3 \end{tikzcd}$$

λ_j minuscule

quiver variety $\mathfrak{M}(\lambda, \mu) = \Phi^{-1}(0) /\!/_\chi \prod GL_{v_i} \to \mathfrak{M}_0(\lambda, \mu)$

affine Gr slice $W^{\underline{\lambda}}_\mu \to W^\lambda_\mu$

Nakajima   irreps of $G^\vee$ geometric Satake

$$H^\bullet(\mathfrak{M}(\lambda, \mu)) \cong (V(\lambda_1) \otimes \cdots \otimes V(\lambda_n))_\mu \cong H^\bullet(W^{\underline{\lambda}}_\mu)$$

Quantum cohomology

$Y \to X$ conical symplectic resolution with Hamiltonian T-action

Quantum cohomology is a ring structure on $H^\bullet_{T\times\mathbb{C}^\times}(Y,\mathbb{C})$
defined using 3 pt GW invariants.

Special features for conical symplectic res.

(i) We need to work with $\mathbb{C}^\times$-equiv cohomology,
otherwise there are no quantum corrections.

(ii) Usually QH is defined over a power series ring $\sum_{\beta \in \text{eff curve classes}} \ldots q^\beta$
but for symplectic resolution we work over
$\mathbb{C}[H^2(Y,\mathbb{C}^\times)^{reg}]$

Conjecture [Okounkov] "Kahler roots"

There exists a finite subset $\Delta_+ \subset H_2(Y,\mathbb{Z})$ and for
each $\alpha \in \Delta_+$, an operator $L_\alpha \in \mathrm{End}(H^\bullet_{T\times\mathbb{C}^\times}(Y))$ s.t.
for $u \in H^2_{T\times\mathbb{C}^\times}(Y)$, $q \in H^2(Y,\mathbb{C}^\times)^{reg} := \{q : q^\alpha \neq 1 \ \forall \alpha \in \Delta_+\}$

$$\underbrace{u *_q -}_{\text{quantum mult}} = \underbrace{u \cdot -}_{\text{classical mult}} + \underbrace{\hbar}_{\mathbb{C}^\times\text{-equiv param}} \sum_{\alpha\in\Delta_+} \langle \alpha, u\rangle \frac{q^\alpha}{1-q^\alpha} L_\alpha(-)$$

an equality of operators on $H^\bullet_{T\times\mathbb{C}^\times}(Y)$

Moreover $L_\alpha \in H_{2d}(\underbrace{Y\times_X Y}_{\text{Steinberg variety}}) = \mathrm{End}(\pi_*\mathbb{C}_Y) \circlearrowright H^\bullet(Y)$

$d = \dim Y = \dim Y \times_X Y$

Moreover, all quantum multiplications (not necc by divisors) are also defined for $q \in H^2(Y, \mathbb{C}^\times)^{reg}$

This conjecture has been verified, and L_α determined in many examples by Okounkov + students/collaborators:

$T^* G/B$	Braverman-Maulik-Okounkov	~2010
quiver varieties	M-O	
hypertoric varieties	McBreen-Shenfeld	
$T^* G/P$	Su	
affine Gr slices	Danilenko	~2020

QH is commutative, so we get

$$H^2(Y, \mathbb{C}^\times)^{reg} \longrightarrow \text{commutative subspaces of } \mathrm{End}(H^\bullet_{T\times\mathbb{C}^\times}(Y))$$

$$q \longmapsto Q(q) = \{u *_q - : u \in H^2_{T\times\mathbb{C}^\times}(Y)\}$$

$$H^2(Y, \mathbb{C}^\times)^{reg} \longrightarrow \text{commutative subalgebras of } \mathrm{End}(H^\bullet_{T\times\mathbb{C}^\times}(Y))$$

$$q \longmapsto QH^\bullet_{T\times\mathbb{C}^\times}(Y)_q$$

Strategy: find an algebra U acting on $H^\bullet_{T\times\mathbb{C}^\times}(Y)$ and look for a commutative subspace/subalg $A \subset U$.