Quantum cohomology of ADE quiver varieties

Theorem

① There is an action of $\mathfrak{g}$ on $\bigoplus_\mu H^\bullet(\mathfrak{M}(\lambda,\mu))$ s.t.

$$H^\bullet(\mathfrak{M}(\lambda,\mu)) \cong (V(\lambda_1)\otimes\cdots\otimes V(\lambda_n))_\mu \qquad \text{[Nakajima]}$$

$\mathfrak{g}$ acts by correspondences

$$U\mathfrak{g} \longrightarrow H_{top}\left(\mathfrak{M}(\lambda)\times_{\mathfrak{M}_0(\lambda)}\mathfrak{M}(\lambda)\right) \qquad \mathfrak{M}(\lambda)=\bigsqcup_\mu \mathfrak{M}(\lambda,\mu)$$

$$\mathfrak{M}_0(\lambda)=\bigcup_\mu \mathfrak{M}(\lambda,\mu)$$

② There is an action of $Y_\mathfrak{g}$ on $\bigoplus_\mu H^\bullet_{T\times\mathbb{C}^\times}(\mathfrak{M}(\lambda,\mu))$

s.t.

$$H^\bullet_{\underline{z},1}(\mathfrak{M}(\lambda,\mu)) = (V(\lambda_1,z_1)\otimes\cdots\otimes V(\lambda_n,z_n))_\mu \qquad \text{[Varagnolo]}$$

$\underline{z}=(z_1,\dots,z_n)$ fundamental $Y_\mathfrak{g}$ modules

We have tautological vector bundles $\mathcal{V}_i$ on $\mathfrak{M}(\lambda,\mu)$

and this gives line bundles $\mathcal{L}_i=\det\mathcal{V}_i$ $\qquad i\in I$

So we get $\quad \mathbb{Z}^I \longrightarrow H^2(\mathfrak{M}(\lambda,\mu),\mathbb{Z})$

$$\varepsilon_i \longmapsto c_1(\mathcal{L}_i)$$

this map is an isomorphism as long as $v_i\neq 0\ \forall i$

so we use it to identify $\mathfrak{h}\cong H^2(\mathfrak{M}(\lambda,\mu),\mathbb{C})$ $\quad \mathfrak{h}^*\cong H_2(\mathfrak{M}(\lambda,\mu),\mathbb{C})$ (*)

Cartan subalg of $\mathfrak{g}$ $\qquad H=H^2(\mathfrak{M}(\lambda,\mu),\mathbb{C}^\times)$

Theorem [Maulik-Okounkov]

$\Delta_+\subset H_2(\mathfrak{M}(\lambda,\mu),\mathbb{Z})\cong$ root lattice of $\mathfrak{g}$

① Under (*) Δ_+ = usual positive roots of $\mathfrak{g}$

$$H^2(\mathfrak{M}(\lambda,\mu),\mathbb{C}^\times)^{reg} = H^{reg} = \{q\in H : q^\alpha\neq 1 \ \ \forall\ \alpha\in\Delta_+\}$$

② For $u \in H^2_{T\times\mathbb{C}^*}(M(\lambda,\mu))$ we have

$$u *_q - = u\cdot - + \hbar \sum_{\alpha\in\Delta_+} \langle u,\alpha\rangle \frac{q^\alpha}{1-q^\alpha} L_\alpha(-)$$

where $L_\alpha = e_\alpha f_\alpha + f_\alpha e_\alpha \in (U\mathfrak{g})_0$ $\quad e_\alpha \in \mathfrak{g}_\alpha \quad f_\alpha \in \mathfrak{g}_{-\alpha}$

"root vectors"

eg $\mathfrak{g}=\mathfrak{sl}_m$ $\quad \alpha = \varepsilon_i - \varepsilon_j = (0 \ldots 1 \ldots -1\ 0 \ldots 0)$ $\quad e_\alpha = e_{ij} : \begin{bmatrix} 0 & & 1 \\ & \ddots & \\ & & 0 \end{bmatrix}$

$H_\bullet(M(\lambda,\mu) \times_{M_0(\lambda,\mu)} M(\lambda,\mu))$

③ For u as above, the entire $u *_q -$ is the image under $(Y\mathfrak{g})_0 \to \mathrm{End}(H^\bullet_{T\times\mathbb{C}^*}(M(\lambda,\mu)))$ $\quad (**)$ of certain elements in $Y\mathfrak{g}$ called "Dynamical Hamiltonians" $\rightsquigarrow$ "trigonometric Casimir Hamiltonians"

<u>Conjecture</u> (Maulik-Okounkov, proven for T^*(partial flag variety) by GRTV)

Fix $q \in H$ The subalgebra $QH^\bullet_{T\times\mathbb{C}^*}(M(\lambda,\mu))_q \subset \mathrm{End}(H_{T\times\mathbb{C}^*}(M(\lambda,\mu)))$ equals the image of Bethe subalgebra $B(q) \subset Y\mathfrak{g}$ under $(**)$

Quantum cohomology of affine Grassmannian slices

$$Gr_G = G(\mathbb{C}[t,t^{-1}])/G(\mathbb{C}[t])$$

Geometric Satake correspondence gives a relation between

$\overline{Gr^\lambda} \subset Gr_{G^\vee}$ and $V(\lambda)$ rep of G

As above $\lambda = \lambda_1 + \dots + \lambda_n$ $\quad \lambda_j$ minuscule dom. coweight $\quad (P,\varphi)$

$W^{\underline{\lambda}}_\mu = Gr^{\lambda_1} \tilde\times \cdots \tilde\times Gr^{\lambda_n} \times_{Gr} W_\mu \longrightarrow W^\lambda_\mu = \overline{Gr^\lambda} \cap W_\mu$ $\quad \varphi : P|_{\mathbb{P}^1\setminus\{0\}} \to P^0|_{\mathbb{P}^1\setminus\{0\}}$

Naive identification: $H^{\bullet}_{\theta,1}(W^{\lambda}_{\mu}) \cong (V(\lambda_1)\otimes\cdots\otimes V(\lambda_n))_{\mu}$ (*)

stable basis $\longleftrightarrow$ tensor product basis

More sophisticated:

Given $\theta\in\mathfrak{h}^*$, let $M(\theta) = U\mathfrak{g}\otimes_{U\mathfrak{b}}\mathbb{C}\theta$ be the Verma module

For any wt μ, any any f.d. rep V

$$\mathrm{Hom}_{\mathfrak{g}}(M(\theta+\mu),\ M(\theta)\otimes V) \longrightarrow V_\mu$$
$$\varphi \longmapsto (v_\theta^*\otimes I)(\varphi(v_{\theta+\mu}))$$

this map is an iso for generic θ

Theorem [Ginzburg-Rice] $V(\underline{\lambda}) = V(\lambda_1)\otimes\cdots\otimes V(\lambda_n)$

There is a natural iso.

$$\mathrm{Hom}_{\mathfrak{g}}(M(\theta+\mu),\ M(\theta)\otimes V(\underline{\lambda})) \cong H^{\bullet}_{\theta,1}(W^{\underline{\lambda}}_{\mu}) \qquad (**)$$

Consider $1\le i\ne j\le n$, $\Omega^{(ij)}\in U\mathfrak{g}^{\otimes n}$ insert Casimir $\Omega\in\mathfrak{g}\otimes\mathfrak{g}$ into i,j tensor factors $\Omega^{(ij)} = \sum_k 1\otimes\cdots\otimes \underset{i}{X_k}\otimes\cdots\otimes\underset{j}{X^k}\otimes\cdots\otimes 1$ $\{X_k\}\,\{X^k\}$ dual bases of $\mathfrak{g}$

$\Omega^{(ij)}$ commutes with diagonal $\mathfrak{g}\subset(U\mathfrak{g})^{\otimes n}$

For $z_1,\dots,z_n\in\mathbb{C}$ distinct, let

$$H_i(\underline{z}) = \sum_{j\ne i}\frac{\Omega^{(ij)}}{z_i-z_j}\in(U\mathfrak{g})^{\otimes n} \qquad \text{Gaudin Hamiltonians}$$

These $H_1(\underline{z}),\dots,H_n(\underline{z})$ commute among themselves and so span a commutative subspace $G(\underline{z})\subset(U\mathfrak{g})^{\otimes n}$

We have $W^{\lambda}_{\mu}\subset Gr^n$ and $H^2(Gr,\mathbb{Z})=\mathbb{Z}$ so we get

$$\begin{tikzcd} H^2(Gr^n) = H^2(Gr)^{\otimes n} = \mathbb{Z}^n \arrow[r] \arrow[dr] & H^2(W^\lambda_\mu) \\ & \mathbb{Z}^n/\mathbb{Z}(1,\dots,1) \arrow[u, "\text{usually an iso.}"'] \end{tikzcd}$$

D_j = image of ε_j

eg. $G = GL_m$ $\quad \lambda = (\omega_{k_1}, \dots, \omega_{k_n})$ $\quad k_i \in \{1,\dots,m-1\}$

$$W^\lambda_\mu = \left\{ \mathbb{C}[t]^m \supset L_1 \supset L_2 \supset \dots \supset L_n : \begin{array}{l} \dim L_j/L_{j+1} = k_j \quad tL_j \subset L_{j+1} \\ L_n \in W_\mu \end{array} \right\}$$

we get line bundles $\det(L_j/L_{j+1})$ on W^λ_μ $\quad D_j = c_1(\det L_j/L_{j+1})$

They generate $\mathrm{Pic}\, W^\lambda_\mu$ but $\det(L_0/L_n)$ is trivial.

$$((U\mathfrak{g})^{\otimes n})^{\mathfrak{g}} \longrightarrow \mathrm{End}_{\mathfrak{g}}(V(\lambda))$$

$$\Omega^{(ij)} \longmapsto H_{top}(Gr^\lambda \times_{Gr} Gr^\lambda)$$

Theorem [Danilenko]

① Under $\mathbb{Z}^n/\mathbb{Z}(1..1) \cong H^2(W^\lambda_\mu, \mathbb{Z})$

$\{(a_1,\dots,a_n) : a_1+\dots+a_n = 0\} = \mathbb{Z}^n_0 \cong H_2(W^\lambda_\mu, \mathbb{Z})$ the roots $\Delta_+ = \{\varepsilon_i - \varepsilon_j : i<j\}$

② Using $H^\bullet_{T\times\mathbb{C}^\times}(W^\lambda_\mu, \mathbb{C}) \cong V(\lambda)_\mu \otimes H^\bullet_{T\times\mathbb{C}^\times}(pt)$ $\quad q^{\varepsilon_i-\varepsilon_j} = q_i q_j^{-1}$

for all $u \in H^2_{T\times\mathbb{C}^\times}(W^\lambda_\mu)$ and $q \in H^2(W^\lambda_\mu, \mathbb{C}^\times)^{reg} = (\mathbb{C}^\times)^n \setminus \Delta / \mathbb{C}^\times$ we have

$$u *_q - = u \cdot - + \hbar \sum_{i<j} \langle u, \varepsilon_i - \varepsilon_j \rangle \frac{q_i q_j^{-1}}{1 - q_i q_j^{-1}} \Omega^{(ij)}(-) \qquad L_{\varepsilon_i-\varepsilon_j} = \Omega^{(ij)}$$

③ Using $H^\bullet_{\theta,1}(W^\lambda_\mu) \cong \mathrm{Hom}_{\mathfrak{g}}(M(\theta+\mu), M(\theta)\otimes V(\lambda))$

for $q \in (\mathbb{C}^\times)^n \setminus \Delta / \mathbb{C}^\times$ we have

$$D_i *_q - = \underset{\cap\ (\hat{U}\mathfrak{g})^{\otimes n+1}}{H_i(0, q_1, \dots, q_n)} \text{ acting on}$$

"quantum multiplication by divisors come from Gaudin Hamiltonians via the Ginzburg-Riche iso"

We can avoid the Verma modules stuff as follows:

$$H^{trig}_{i,\theta}(q_1,\dots,q_n) = \theta^{(i)} + \sum_j \Omega^{(ij)}_- + \sum_{j\neq i} \frac{q_i \Omega^{(ij)}}{q_i - q_j} \in (U\mathfrak{g})^{\otimes n}$$

③′ $\quad D_i *_q - = H_{i\,\theta}^{trig}(q)$ acting on $V(\lambda)_\mu$

actually ③′ appears in Danilenko, equivalence of ③ and ③′ is proven in [Ilin-K-Rybnikov]

There is a maximal commutative subalgebra $A(z_1, \dots, z_n) \subset (U\mathfrak{g})^{\otimes n}$ called the Gaudin algebra, containing $H_1(\underline{z}), \dots H_n(\underline{z})$

It was defined by [Feigin-Frenkel-Reshetikhin] using the centre of $U\hat{\mathfrak{g}}$ at critical level.

Conjecture

Under $\mathrm{Hom}_{\mathfrak{g}}(M(\theta+\mu), M(\theta) \otimes V(\lambda)) \cong H^\bullet_{\theta,1}(W^\lambda_\mu)$

action of $A(0, q_1, \dots, q_n)$ = action of $QH^\bullet_{\theta,1}(W^\lambda_\mu)_q$

$Y \quad y$

$\{0\} \in \mathcal{B} = H^2(Y)$

Summary $\qquad \lambda = \lambda_1 + \dots + \lambda_n \quad \mu$

Analysis of $u *_q - = u \cdot - + \hbar \sum_{\alpha \in \Delta_+} \langle \alpha, \mu \rangle \frac{q^\alpha}{1-q^\alpha} L_\alpha(-) \qquad q \in H^2(Y, \mathbb{C}^\times)^{reg}$

	Quiver variety $\mathfrak{M}(\lambda,\mu)$	Affine Gr slice W^λ_μ
Acting torus T	$(\mathbb{C}^\times)^n \subset \prod GL(W_i) / \mathbb{C}^\times$	$T \subset G$
Kähler torus $H^2(Y, \mathbb{C}^\times)$	$H \subset G$ max torus	$(\mathbb{C}^\times)^n / \mathbb{C}^\times$
Kahler roots	Δ_+ usual pos. roots	$\{\varepsilon_i - \varepsilon_j : i<j\}$ "A_{n-1} root system"
Steinberg operator	$L_\alpha = e_\alpha f_\alpha + f_\alpha e_\alpha \in U\mathfrak{g}$	$L_{ij} = \Omega^{(ij)} \in (U\mathfrak{g})^{\otimes n}$
quantum mult	dynamical Hamiltonian in $Y\mathfrak{g}$	Gaudin Hamiltonian $H_i(0, q_1, \dots, q_n)$ acting via Ginzburg-Riche
quantum cohomology algebra Conjectural	image of Bethe algebra in $Y\mathfrak{g}$	image of Gaudin algebra acting via Ginzburg-Riche

duality