1. Stable log maps

As in stable maps, but in log category.

Target: $X = (\underline{X}, \mathcal{M}_X)$ fs log scheme

later: X log-smooth & projective / $B = (\underline{B}, \mathcal{M}_B)$

Stable log map over a log point:

$$\begin{array}{ccc} C = (\underline{C}, \mathcal{M}_C) & \xrightarrow{f} & X \\ \pi \downarrow & & \\ W = \mathrm{Spec}\,(Q \to \mathbb{C}) & & \end{array}$$

reduced, connected

log smooth, integral

Note: We need $Q \neq 0$ to allow nodal domains

Stability: $\underline{C} \xrightarrow{\underline{f}} \underline{X}$ stable (i.e. $\mathrm{Aut}(\underline{C}/\underline{X})$ finite)

Pblm: Can't give a good theory since $\{$possible Q's for fixed $\underline{C} \to \underline{X}\}$ is not bounded: We can e.g. always add factors of $\mathbb{N}$:

$$\cdots \leftarrow Q \oplus \mathbb{N}^2 \leftarrow Q \oplus \mathbb{N} \leftarrow Q$$

Common in log moduli problems: We need to connect the log structure of the parameter space to the geometry of the family, i.e. we need a universal choice of Q (as for nodal curves, where $Q = \mathbb{N}^{\#\text{nodes}}$ is universal)

Solution for stable log maps: $Q = \mathrm{Hom}_{\mathrm{mon.}}\left((\text{tropical moduli})_{\mathbb{Z}}, \mathbb{N}\right)$

$\rightsquigarrow$ notion of <u>basic</u> stable log maps.

2. Basic stable log maps

Tropicalization of a stable log map $\begin{array}{c} C \xrightarrow{f} X \\ \downarrow \\ W = \mathrm{Spec}(Q \to \mathbb{C}) \end{array}$ over a log point has a type:

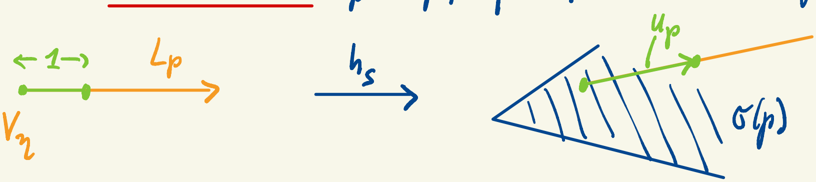
$$\mathrm{Trop}(C/W, f) = \begin{array}{c} \Sigma_C \xrightarrow[\;h\;]{\mathrm{Trop}(f) = } \Sigma_X \\ \downarrow \\ \tau = Q^\vee_{\mathbb{R}} \end{array} \quad \rightsquigarrow \forall s \in \tau : \quad \boxed{\Gamma_s \xrightarrow{h_s} \Sigma_X} \begin{array}{c} \\ \downarrow \\ \{s\} \end{array}$$

tropical stable map

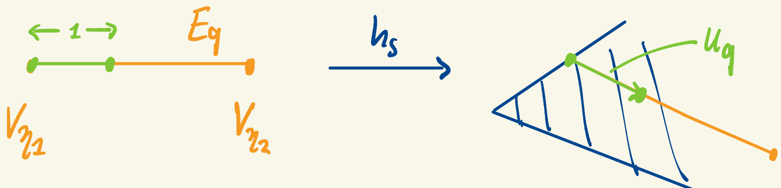
The $h_s : \Gamma_s \to \Sigma_X$ have the same type $\forall s \in \mathrm{Int}(\tau)$.

type:
- Γ_s as an abstract graph $\rightsquigarrow$ vertices V_η, edges E_q, legs L_p
- the smallest cones $\sigma(\eta), \sigma(q), \sigma(p)$ of Σ_X containing $h_s(V_\eta), h_s(E_q), h_s(L_p)$
- contact orders $u_p \in \sigma(p)$, $u_q \in \sigma(q)^{gp}$ for marked pts p, nodes q.

u_p:



u_q:

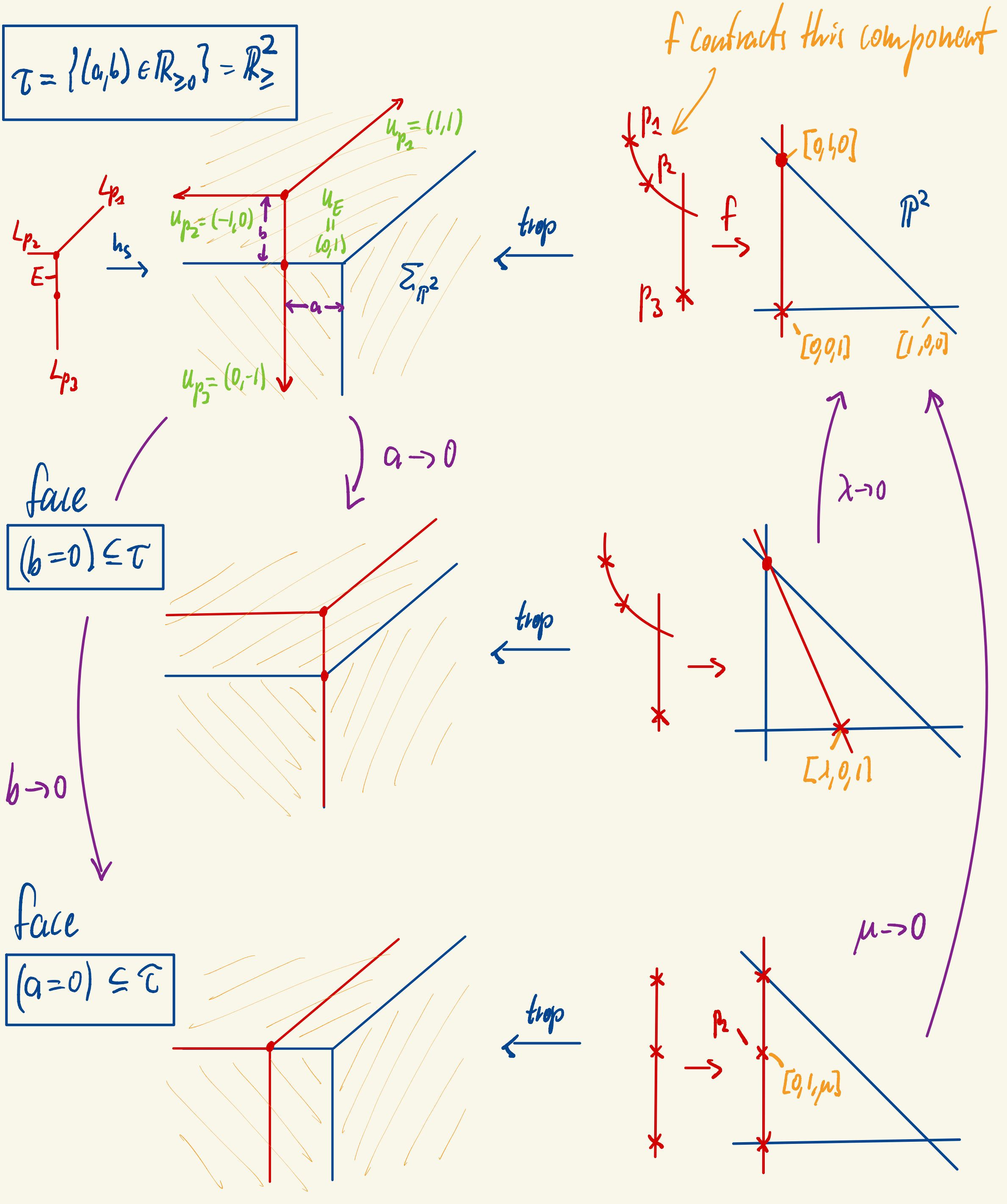


[sign of u_q depends on orientation of E_q]

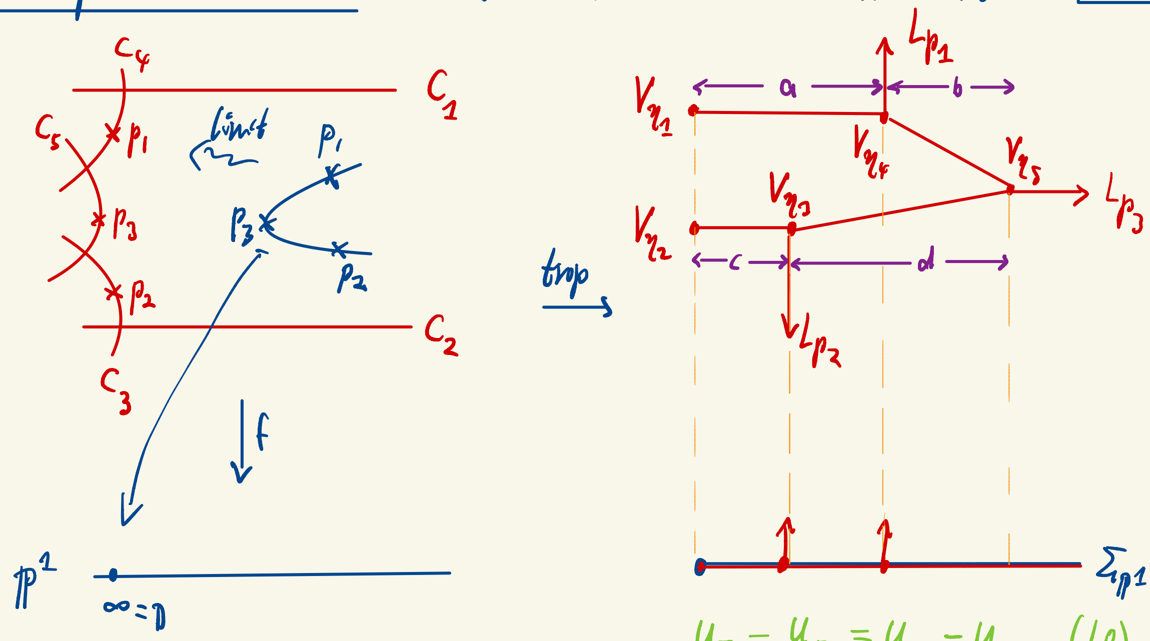
(Local) tropical moduli of type $(\Gamma, \underline{\sigma}, \underline{u})$ $\rightsquigarrow$ cone τ parametrizing tropical stable maps of this type and their limits.

Expl: a) $X = \mathbb{P}^2$, deg $= 1$ stable log maps, three marked points.

$$\boxed{\tau = \{(a,b) \in \mathbb{R}_{\geq 0}\} = \mathbb{R}^2_{\geq}}$$



b) Expl. with $\tau \neq \mathbb{R}^k_{\geq 0}$: $\quad X = (\mathbb{P}^1, D = \{\infty\}) \implies \Sigma_X = \mathbb{R}_{\geq 0}$



$u_{E_1} = u_{E_2} = u_{E_3} = u_{E_4} = (1,0)$

Result: $\boxed{\tau = \{(a,b,c,d) \in \mathbb{R}^4_{\geq 0} \mid a+b = c+d\}}$

Def:
$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d] & X \\ \operatorname{Spec}(Q \to \mathbb{C}) & \end{tikzcd} \quad \text{is basic} \iff \quad \begin{tikzcd} \Sigma_C \arrow[r] \arrow[d] & \Sigma_X \\ \operatorname{Hom}(Q, \mathbb{R}_{\geq 0}) & \end{tikzcd}$$
is universal tropical stable map (of some type)

In families: Require basicness at every geometric point of the base.

3. Moduli space

<u>Thm</u> (Abramovich/Chen, Gross/S.)

There exists a "good" moduli space $\mathcal{M}(X)$ of basic stable log maps:

(i) is a separated Deligne-Mumford, locally of finite type/$\mathbb{C}$, with an fs log structure (stalks of $\overline{\mathcal{M}}$ = basic monoids)

(ii) fulfills the valuative criterion of properness

(iii) Assuming X log smooth over some log scheme B, $\mathcal{M}(X)$ has a perfect obstruction theory in the sense of Behrend-Fantechi relative $\mathfrak{M}$ = stack of log-smooth curves (with any fs-log structure on the base)

(iv) is proper/B when fixing topological data β

$[\beta = (g, \underline{u}, A \in H_2(X))$, g genus, $\underline{u}$ contact orders, A curve class$]$

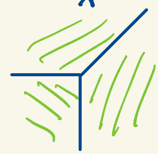
<u>Cor:</u> (iii) provides a virtual fundamental class $[\mathcal{M}(X)]_{virt}$ on $\mathcal{M}(X)$ $\rightsquigarrow$ log GW-invts by pairing with cohomology classes on X via the evaluation maps at the marked pts.

Expl of some $\mathcal{M}(X,\beta)$:

$X = \mathbb{P}^2$, $\beta = (g=0,\ \underline{u} = (u_1,u_2,u_3),\ A = [\text{line}])$

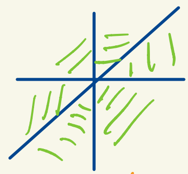
$u_1 = (-1,0)$, $u_2 = (0,-1)$, $u_3 = (1,1)$

Σ_X



Result:

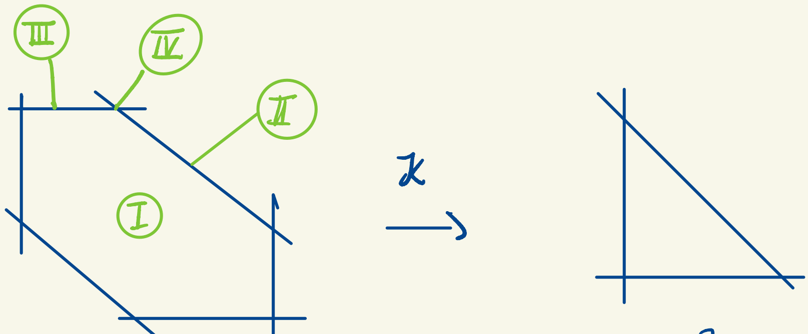
$\mathcal{M}(X,\beta) = Bl_3(\mathbb{P}^2) = TV(\Sigma_3)$, $\Sigma_3 =$



$$\begin{tikzcd}
\mathcal{M}(X,\beta) \arrow[d, "Bl_3"', "\kappa"] \arrow[dr, "\iota"] & \\
(\mathbb{P}^2)^{*} & \mathcal{M}(\underline{X}, (g=0, 3 \text{ marked pts}, [\text{line}])) \arrow[l, "\pi"'] \\
f(\mathbb{P}^1) & (f:\mathbb{P}^1 \to \mathbb{P}^2, \{x_1,x_2,x_3\} \subseteq \mathbb{P}^1) \arrow[l, mapsto]
\end{tikzcd}$$

— ordinary stable map space

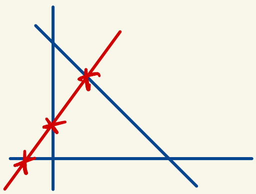
- π is an $S^3\mathbb{P}^1 = \mathbb{P}^3$-fibration
- ι is a closed embedding — [gen'ly, the map from $\mathcal{M}(X)$ to the ordinary stable map space is always finite.]
- κ is birational



$Bl_3\mathbb{P}^2 = \mathcal{M}(X,\beta)$ $\qquad\qquad$ $\mathbb{P}^2$

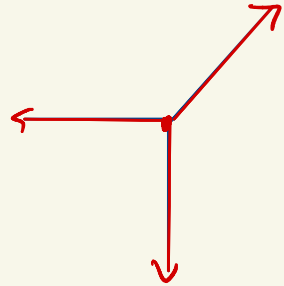
Zoo of some log strata of $\mathcal{M}(X,\beta)$:

I)

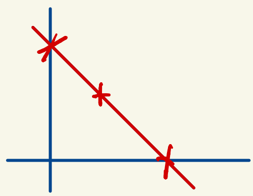


$C \xrightarrow{f} \mathbb{P}^2$
strict

$\xrightarrow{\text{trop}}$

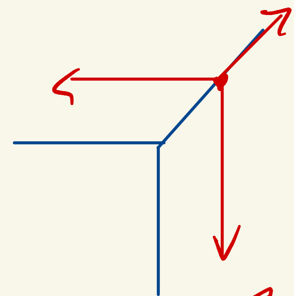


II)

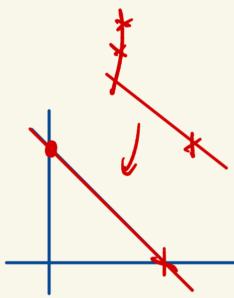


$C \xrightarrow{f} \mathbb{P}^2$
strict away from the p_i

$\xrightarrow{\text{trop}}$

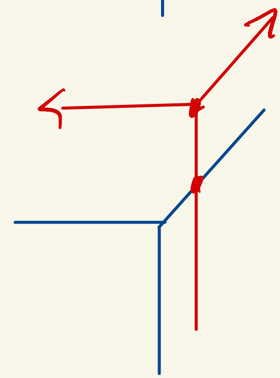


IV)

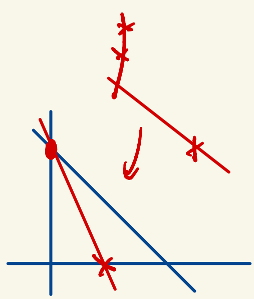


$C \xrightarrow{f} \mathbb{P}^2$
contracts one irred. cmp

$\xrightarrow{\text{trop}}$

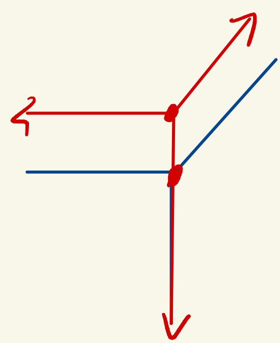


III)



$C \xrightarrow{f} \mathbb{P}^2$
contracts one irred. cmp

$\xrightarrow{\text{trop}}$



3. Artin fans

$[pt/\mathbb{G}_m^n] = B\mathbb{G}_m^n$

$$P = \sigma^\vee \cap M \rightsquigarrow \mathcal{A}_\sigma = [TV(\sigma)/\mathbb{G}_m^n] \text{ algebraic stack}$$

$\mathcal{A}_\sigma$ classifies $\mathbb{G}_m^n$-bundles together with an $\mathbb{G}_m^n$-equivariant morphism to $TV(\sigma)$:

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & TV(\sigma) \arrow[d, "/\mathbb{G}_m^n"] \\ W \arrow[r] & \mathcal{A}_\sigma \end{tikzcd}$$

$\tau \leq \sigma$ face $\rightsquigarrow$ <u>open</u> embedding $\mathcal{A}_\tau \hookrightarrow \mathcal{A}_\sigma$: $\tau_{\mathbb{Z}}^\vee = (\sigma_{\mathbb{Z}}^\vee + \tau_{\mathbb{Z}}^\perp)/\tau_{\mathbb{Z}}^\perp$

e.g. $[\mathbb{A}^1/\mathbb{G}_m] = [\mathbb{A}^1 \times \mathbb{G}_m/\mathbb{G}_m^2] \hookrightarrow [\mathbb{A}^2/\mathbb{G}_m^2]$

Now $X = (\underline{X}, \mathcal{M}_X)$ fine log scheme $\rightsquigarrow$ diagram of cones Σ_X

<u>Artin fan</u> of X: $\boxed{\mathcal{X} := \varinjlim_{\sigma \in \Sigma_X} \mathcal{A}_\sigma}$ is an algebraic stack

<u>Use</u>: $\mathcal{X}$ algebraizes the tropical geometry in Σ_X:

<u>Prop</u>: Assume $\mathcal{A}_X$ has a Zariski covering by $\mathcal{A}_\sigma$'s. [e.g. X is a Zariski log scheme, log smooth/$\mathbb{C}$]

Then $\forall$ fs log scheme T,

$$\boxed{\mathrm{Hom}_{fs}(T, \mathcal{X}) = \mathrm{Hom}_{cones}(\Sigma_T, \Sigma_X)}$$

canonical bijection

Application: stable log map into $\mathcal{X}$:

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \mathcal{X} \\ \operatorname{Spec}(Q \to \mathbb{C}) & \end{tikzcd}$$

is the same as

- the log smooth curve $C \to \operatorname{Spec}(Q \to \mathbb{C})$
- and a tropical stable map

$$\begin{tikzcd} \Sigma_C \arrow[r] \arrow[d] & \Sigma_{\mathcal{X}} = \Sigma_X \\ Q^\vee_{\mathbb{R}} & \end{tikzcd}$$

still makes sense!

Thus $\mathfrak{M}(\mathcal{X}) = ($stack of basic stable log maps into $\mathcal{X})$ is algebraic!

Note: The forgetful map $\mathfrak{M}(\mathcal{X}) \to \mathfrak{M}$ is strict and étale, and we have the factorization

$$\mathcal{M}(X) \to \mathfrak{M}(\mathcal{X}) \to \mathfrak{M}.$$

Thus $\mathfrak{M}(\mathcal{X})$ is a discrete (tropical) refinement of the stack $\mathfrak{M}$ of (logarithmic) domain curves, which itself is a discrete refinement of the stack of nodal curves (w/o lo structure).

$\boxed{\mathfrak{M}(\mathcal{X}) \text{ is the correct stack of domain curves in log-GW theory!}}$

4. $\mathfrak{M}(\mathcal{X},\tau)$ and log-GW of restricted type

τ type of tropical stable map to $\Sigma_{\mathcal{X}} = \Sigma_X$.
Moduli space of tropical stable maps of type τ is a cone, also denoted τ

Marking by τ of a stable log map is an identification of τ with a face of $Q_{\mathbb{R}}^\vee$:

$$\begin{array}{ccc} C & \xrightarrow{f} & X \text{, or } \mathcal{X} \\ \downarrow & & \\ \mathrm{Spec}(Q \to \mathbb{R}) & & \end{array} \quad \overset{\text{trop}}{\rightsquigarrow} \quad \begin{array}{ccc} \Sigma_C & \longrightarrow & \Sigma_X \\ \downarrow & & \\ Q_{\mathbb{R}}^\vee & & \end{array}$$

Defines closed algebraic substacks $\mathfrak{M}(\mathcal{X},\tau) \subseteq \mathfrak{M}(\mathcal{X})$

$$\mathfrak{M}(\mathcal{X},\tau) \times_{\mathfrak{M}(\mathcal{X})} \mathscr{M}(X) = \mathscr{M}(X,\tau) \subseteq \mathscr{M}(X)$$

Important:
- $\mathfrak{M}(\mathcal{X},\tau)$ is pure-dimensional [unlike $\mathfrak{M}(\mathcal{X})$!]
  $(\dim = 3g-3+k - \dim\tau + \dim B)$
- $\mathscr{M}(X) \xrightarrow{\varepsilon} \mathfrak{M}(\mathcal{X})$ is virtually smooth

  $\Rightarrow \mathscr{M}(X,\tau) \longrightarrow \mathfrak{M}(\mathcal{X},\tau)$ is also

Refined virtual fundamental class: $[\mathscr{M}(X,\tau)]_{virt} = \varepsilon^! [\mathfrak{M}(\mathcal{X},\tau)]_{virt}$

This provides enumerative meaning to any type of tropical stable map!