4) Quivers and curves in higher dimension

$$\begin{array}{ccl} & \nearrow & \text{Log curves in toric varieties [B-Argüz 2302.02068]} \\ \text{Quiver DT invariant} & & \\ & \searrow & \text{Log curves in cluster varieties [B-Argüz 2308.07270]} \\ \downarrow & & \\ \text{Geometric DT for } D^b\mathrm{Coh} & & \end{array}$$

Early works:
- Gross-Pandharipande-Siebert ~ 2009
- "Kronecker/quiver correspondence" [Reineke-Weist, Reineke-Stoppa-Weist]

(Q, W) $\quad N = \mathbb{Z}^d$ $\quad M = \mathrm{Hom}(N, \mathbb{Z})$

$|Q_0| = d$ $\quad M_{\mathbb{R}} = M \otimes \mathbb{R} = \mathrm{Hom}(N, \mathbb{R}) \simeq \mathbb{R}^d$

$$\left.\begin{array}{l}\text{Dimension vector } \gamma \in N \\ \text{Stability parameter } \theta \in \gamma^{\perp} \subset M_{\mathbb{R}}\end{array}\right\}$$

$$\rightarrow \Omega_\gamma^\theta \in \mathbb{Z} \quad \text{DT invariants}$$

Attractor points $\theta_\gamma = \iota_\gamma \omega = \omega(\gamma, -)$ $\qquad \omega : N \times N \to \mathbb{Z}$

$\omega(e_i, e_j) = a_{ij} - a_{ji}$

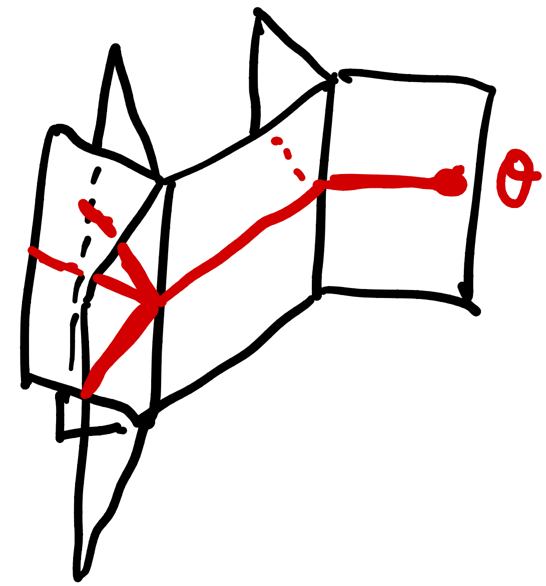
Attractor DT invariants

$$\Omega_\gamma^* = \Omega_\gamma^{\theta_\gamma}$$

Wall-crossing formula $\Rightarrow$:

$$\bar\Omega^\theta_\gamma = \sum_{\gamma=\gamma_1+\dots+\gamma_r} \frac{1}{|\mathrm{Aut}(\{\gamma_j\})|} F^\theta_r(\gamma_1,\dots,\gamma_r) \prod_{j=1}^{r} \bar\Omega^*_{\gamma_j}$$

$$F^\theta_r(\gamma_1,\dots,\gamma_r) = \sum_{T}$$

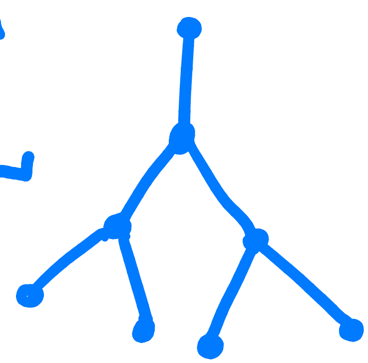


Attractor trees

$$F^\theta_{r,T}(\gamma_1,\dots,\gamma_r)$$

Flow tree formula

$$F^\theta_r(\gamma_1,\dots,\gamma_r) = \sum_{\tilde T}$$



$$\prod_v \varepsilon_v\, \omega(f_{v'}, f_{v''})$$

$$F_r^\theta(\gamma_1,\ldots,\gamma_r) = \sum_T F^\theta_{r,T}(\gamma_1,\ldots,\gamma_r)$$

Attractor Trees $\sim$ Tropical curves in $M_{\mathbb{R}}$
(Trees genus = 0)

GOAL:

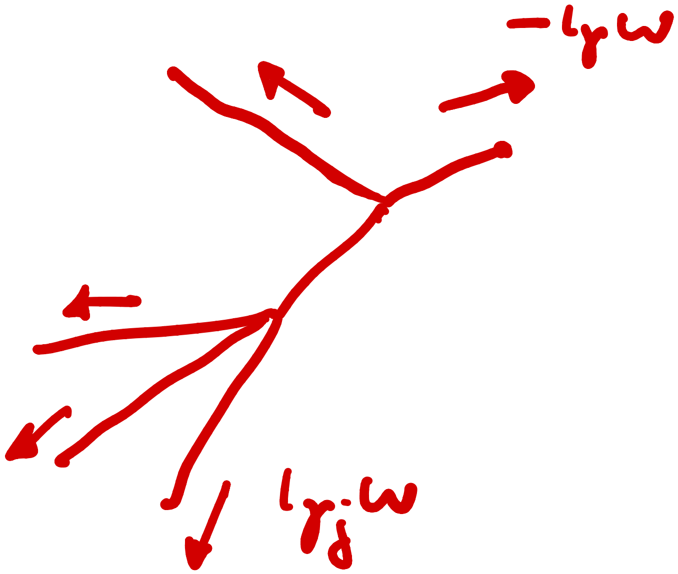
$$F_r^\theta(\gamma_1,\ldots,\gamma_r) = \Bigg| \begin{array}{l}\#\text{ rational log curves} \\ \text{in a toric variety}\end{array}$$

Which toric variety?

$T \subset M_{\mathbb{R}} = \mathrm{Hom}(N, \mathbb{R}) \simeq \mathbb{R}^d$

Fix $\gamma_1, \dots, \gamma_r \in N$

$\gamma = \sum_j \gamma_j$



Fix Σ smooth complete fan in $M_{\mathbb{R}} \simeq \mathbb{R}^d$ containing the rays

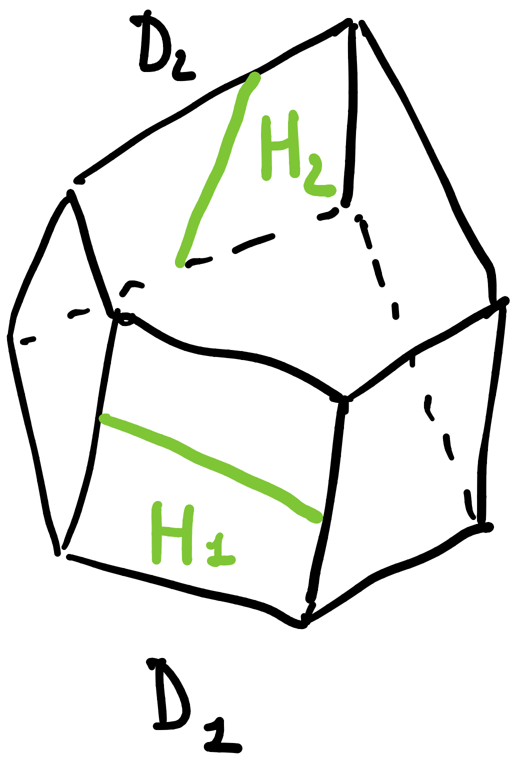
$$\mathbb{R}_{\geq 0}(\iota_{\gamma_j} \omega)$$

$\rightarrow X_\Sigma$ smooth projective toric variety of dim d

$\partial X_\Sigma = X_\Sigma \setminus (\mathbb{C}^*)^d$ Toric boundary

$j = 1, \dots, r$

Ray $\mathbb{R}_{\geq 0}(\iota_{\gamma_j}\omega) \to$ Toric divisor $D_j \subset \partial X_\Sigma$

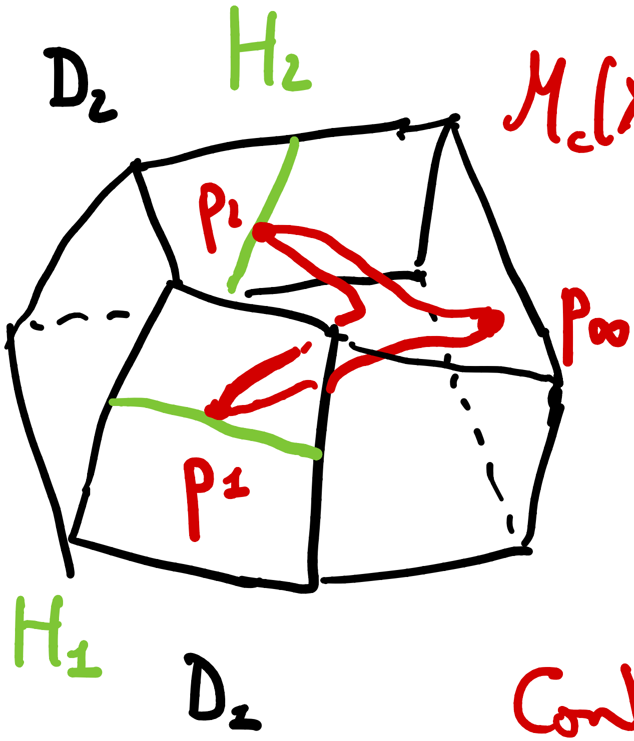


Fix $$H_j \underset{\text{codim } 1}{\subset} D_j \underset{\text{codim } 1}{\subset} X_\Sigma$$

$$H_j = \left\{ z^{\gamma_{j,\mathrm{prim}}}\Big|_{D_j^\circ} = c_j \right\}, \quad c_j \in \mathbb{C}^*$$

$$(\iota_{\gamma_j}\omega)(\gamma_j) = \omega(\gamma_j, \gamma_j) = 0$$

$\Rightarrow$ $z^{\gamma_{j,\mathrm{prim}}}$ regular function on D_j



$$\mathcal{M}_c(X_\Sigma) = \left\{ f : (C, p_1, \dots, p_r, p_\infty) \longrightarrow X_\Sigma \;\middle|\; \begin{array}{l} \bullet \text{ genus } 0 \text{ stable map,} \\ \bullet\ f(p_1) \in H_1, \dots, f(p_r) \in H_r \\ \bullet\ f(C) \cap \partial X_\Sigma = \{ f(p_1), \dots, f(p_r), f(p_\infty) \} \\ \text{Contact orders } l_{\gamma_1} w, \dots, l_{\gamma_r} w, -l_\gamma w \end{array} \right\}$$

Non-compact in general (if $d \geq 3$)

Log geometry!

$$\overline{\mathcal{M}}_c(X_\Sigma) = \{ f : (C, p_1, \dots, p_r, p_\infty) \to X_\Sigma$$

genus 0 stable log maps with contact

orders $\iota_{\gamma_1} w, \dots, \iota_{\gamma_r} w, -\iota_\gamma w$ along ∂X_Σ

such that $f(p_1) \in H_1, \dots, f(p_r) \in H_r \}$

$$\cup$$

$$\mathcal{M}_c(X_\Sigma)$$

General theory of stable log maps

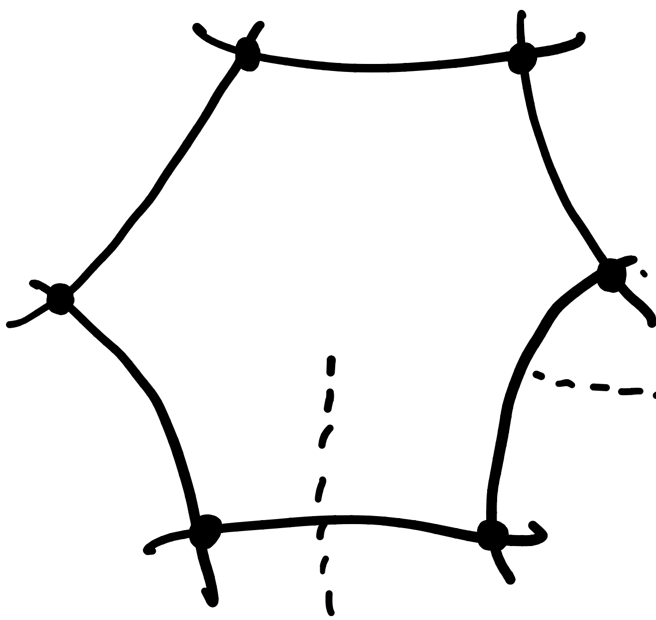
[Gross-Siebert, Abramovich-Chen]

$\Rightarrow \overline{\mathcal{M}}_c(X_\Sigma)$ proper Deligne-Mumford stack

Thm (AB) For general choices of $(c_j)_{1 \leq j \leq r} \in (\mathbb{C}^*)^r$,
$\overline{\mathcal{M}}_c(X_\Sigma)$ is log smooth (toroidal) of dimension $d-2$.

Log structure

Stratification by tropical types



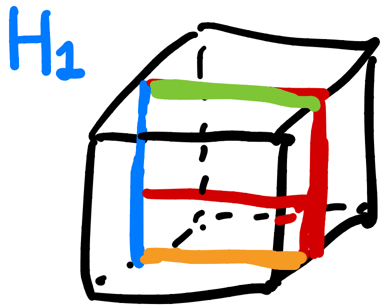
$$\overline{\mathcal{M}}_c(X_\Sigma) = \{ f : C \to X_\Sigma \}$$

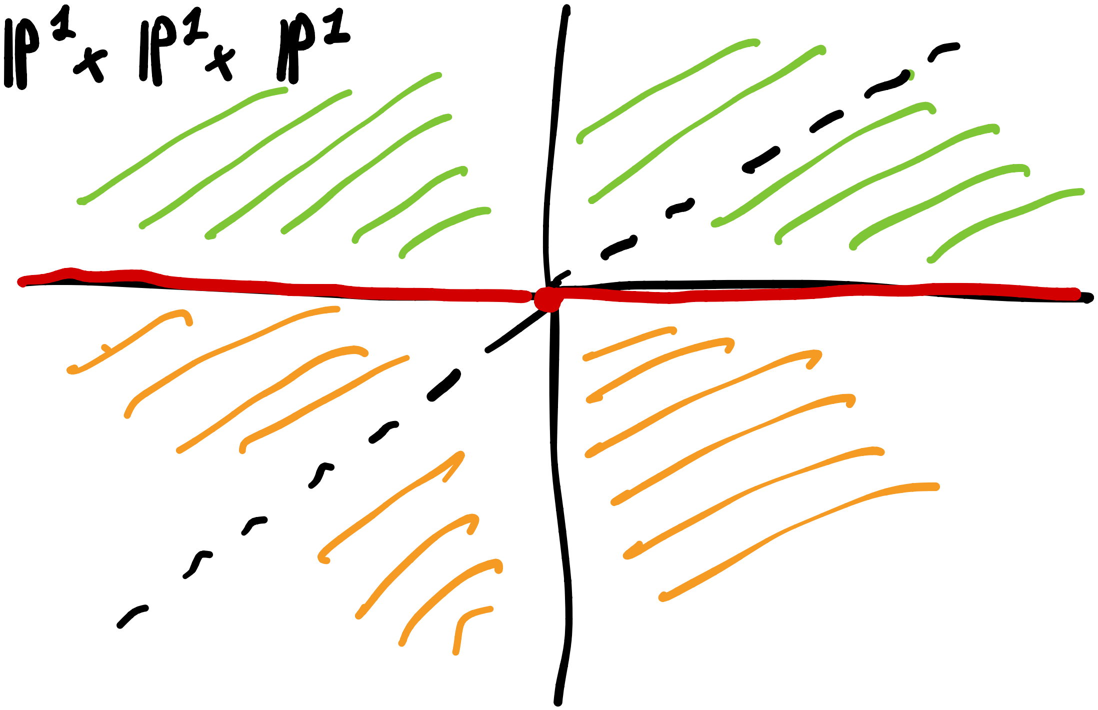
Trop(f) = 2-dim family of tropical curves

Trop(f) = 1-dim family of tropical curves

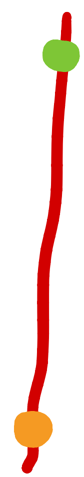
Trop(f) = rigid tropical curve

Example: $X_\Sigma = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$





$$\overline{\mathcal{M}}_c(X_\Sigma) \cong \mathbb{P}^1 =$$



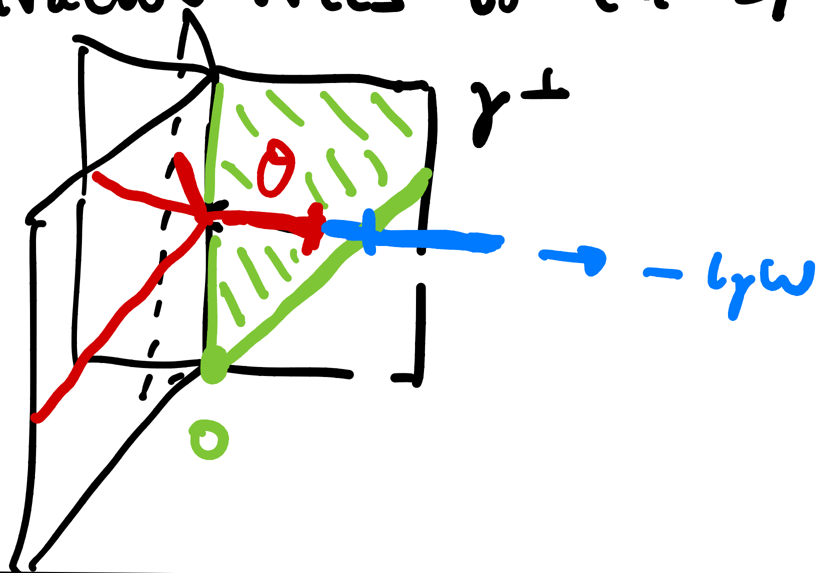
Fix ρ a $(d-2)$-dimensional type of tropical curves

$$GW_\rho(X_\Sigma) = \deg\left[\overline{\mathcal{M}}_c(X_\Sigma, \rho)\right] \in \mathbb{Q}$$

$$\overline{\mathcal{M}}_c(X_\Sigma, \rho) = \bigcup \text{ strata of } \overline{\mathcal{M}}_c(X_\Sigma) \text{ of type } \rho \;\Big|\; \underline{0\text{-dimensional}}$$

"Log Gromov-Witten invariants"

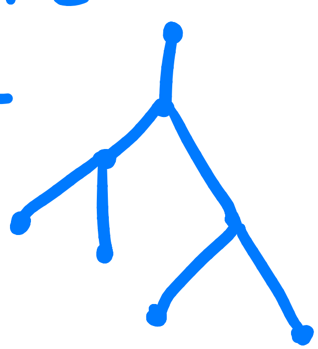
From attractor trees to $(d-2)$-dim tropical types:



$T \to \rho_T$

Thm (AB) $F_r^\theta(\gamma_1, \dots, \gamma_r) = \sum_T GW_{\rho_T}(X_\Sigma)$

Proof: Flow tree formula $\parallel$



// ← Degeneration / Gluing in log GW theory [see Nishinou-Siebert]

$\forall (Q,W)$

$$\bar{\Omega}^{\theta}_{\gamma} = \sum_{\gamma_1+\cdots+\gamma_r=\gamma} \frac{1}{|\mathrm{Aut}(\{\gamma_j\})|} \underbrace{F^{\theta}_{r}(\gamma_1,\dots,\gamma_r)}_{\text{log GW of a toric variety}} \prod_{j=1}^{r} \bar{\Omega}^{*}_{\gamma_j}$$

?

<u>Assume</u> Trivial Attractor DT invariants $\Omega^{*}_{\gamma_j}=0$

$\gamma \notin \mathrm{Ker}\,\omega$

unless $\gamma_j = e_k$

$\Omega^{*}_{e_k} = 1$

$$\bar{\Omega}^{\theta}_{\gamma} = \sum_{\substack{\gamma=\gamma_1+\cdots+\gamma_r \\ \gamma_j^{\text{prim}} \in \{e_1,\dots,e_r\}}} \frac{1}{|\mathrm{Aut}(\{\gamma_j\})|} F^{\theta}_{r}(\gamma_1,\dots,\gamma_r) \prod_{j=1}^{r} \frac{1}{|\gamma_j|^2}$$

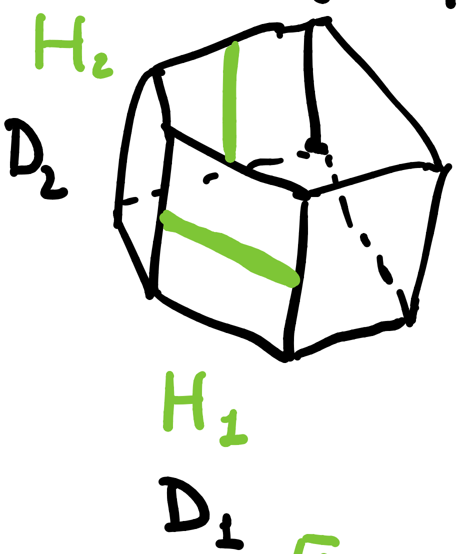
log GW of a Cluster Variety

X_Σ only depends on Q

<u>Rays</u> $\ell_{e_i;w}$

$D_i \subset X_\Sigma$

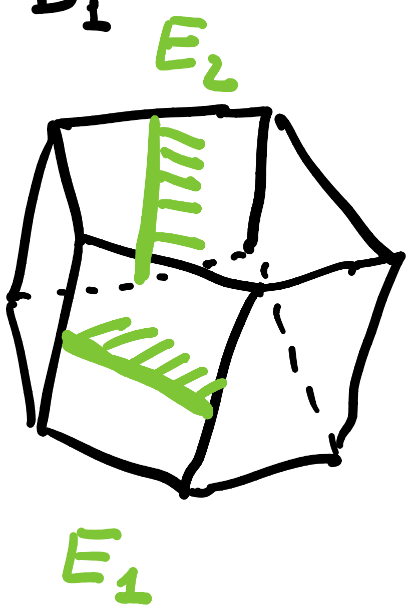
$H_i = \{z^{e_i} = c_i\} \subset D_i$



$$\begin{array}{c} X := \text{Blow-up of } X_\Sigma \text{ along } H_1, \dots, H_n \\ \downarrow \\ X_\Sigma \end{array}$$

$\partial X :=$ strict transform of ∂X_Σ

X



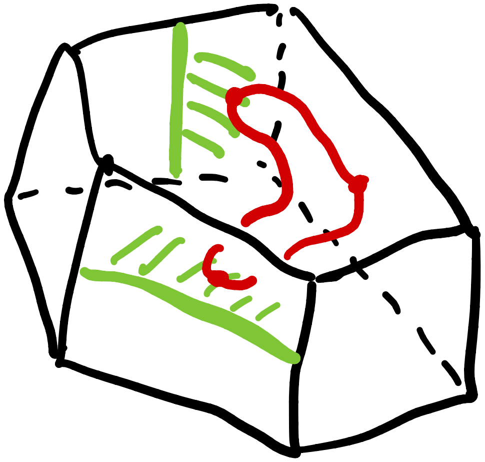
<u>Remark</u> (Gross-Hacking-Keel)

$U := X \setminus \partial X$

$= \mathcal{X}$-cluster variety determined by Q

<u>Poisson</u> $(\leftarrow \omega)$

$= \bigcup (\mathbb{C}^*)^d$



$\mathbb{A}^1$-curves Only one contact point with ∂X

ρ $(d-2)$-dim type

$$\hookrightarrow GW_\rho(X) = \int_{[\overline{M}(X,\rho)]^{vir}} 1 \quad \in \mathbb{Q}$$

[Gross-Siebert, wall functions for wall structures]

Thm [AB]

Assume (Q,W) has trivial attractor DT #. Then $\forall \gamma \notin K_{er\,\omega}$, $\forall \theta \in \gamma^\perp$

$$\overline{\Omega}_\gamma^\theta = \sum_T GW_{\rho_T}(X)$$

[Argüz-Gross HDTV]

Proof: Stability scattering diagram [Bridgeland] = Cluster scattering diagram [GHKK] = Canonical scattering diagram [Gross-Siebert].

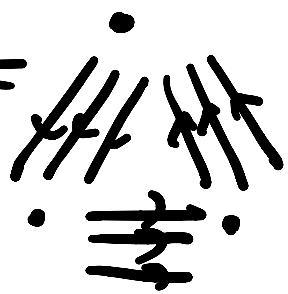
Application: $K_{\mathbb{P}^2}$ Gieseker semistable sheaves

$(r, d, \chi) \in \mathbb{Z}^3$

$\hookrightarrow$ Rank r on $\mathbb{P}^2$

$\hookrightarrow \Omega_{(r,d,\chi)} \in \mathbb{Z}$

<u>If</u> $-1 < \frac{d}{r} \leq 0$, $\exists\, \gamma, \theta$ s.t.

$Q =$ 

W

$$\Omega_{(r,d,\chi)} = \Omega_\gamma^\theta$$

$\Omega_{(r,d,\chi)}$ $\uparrow$ Geometric DT #

Ω_γ^θ $\uparrow$ Quiver DT #

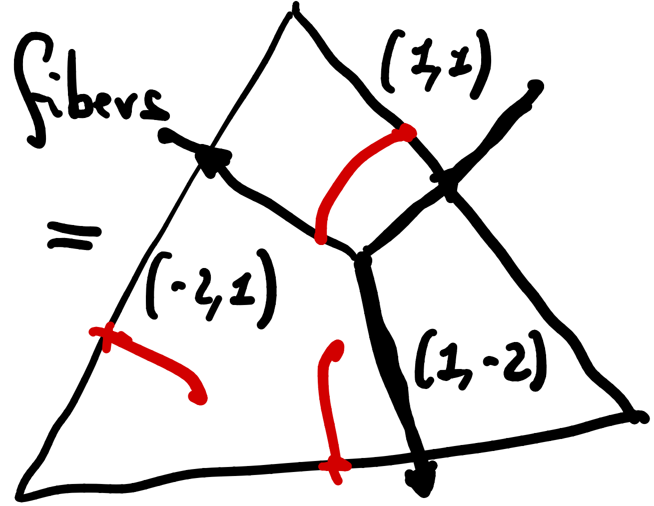
W $\uparrow$ Trivial attractor DT #

[B-Descombes-Le Floch Pioline]

X : $\mathcal{X}$-cluster variety of Q

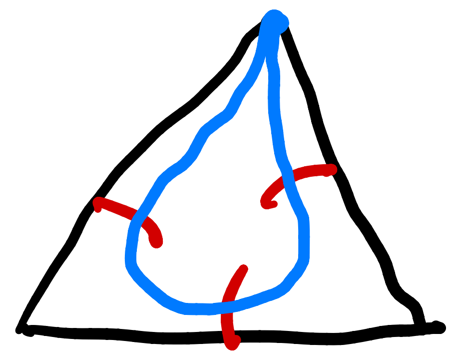
$\downarrow$

$\mathbb{C}^{*}$

Symplectic fibers $X_t =$ 

Thm [AB]

(r,d,χ) $\quad -1 < \frac{d}{r} \leq 0$

$$\Omega_{(r,d,\chi)} = \log \overset{g=0}{GW} \text{ of }$$ 

Not MNOP!

Heuristic: dim reduction + HK rotation

$$\text{DT} \xleftrightarrow{\text{Mirror}} \text{SLAG} \longleftrightarrow \text{Holomorphic}$$

$D^b Coh(K_{\mathbb{P}^2})$ — in mirror of $K_{\mathbb{P}^2}$ — Curves in $U_t = X_t \setminus \partial X_t$