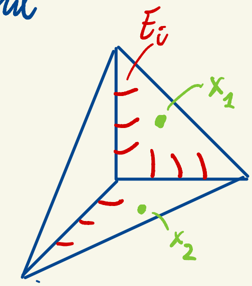
(w/ Abramovich, Chen, Gross)

1. Rigid tropical curves and virtual decomposition

Expl: $X' = V(t f_3(z_0, \dots, z_3) + z_0 \cdots z_3) \subseteq \mathbb{A}^1_t \times \mathbb{P}^3_{z_0 \dots z_3}$

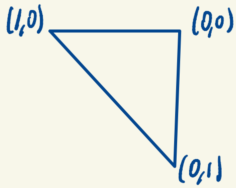
f_3 homog., deg 3, general

$$\begin{array}{c} X \\ \text{resolve} \to \downarrow \\ X' \\ \downarrow \\ \mathbb{A}^1 \end{array} \quad \text{nc degen.}$$

$X_0 =$ 

$= \bigcup_3 Bl_{\dots} \mathbb{P}^2$

$X_t, t \neq 0$: cubic surface, $\simeq Bl_6 \mathbb{P}^2$

$\Sigma(X) = \Sigma(X_0) =$ cone over 

$\simeq \mathbb{R}^3_{\geq 0}$.

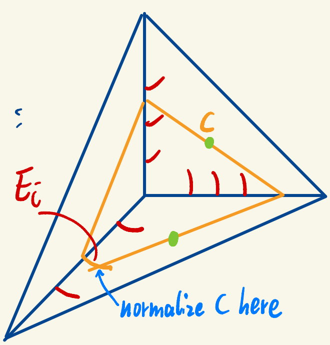
Classical result: There a 12 nodal cubics in X_t through two general pts.

Reproduce by stable log maps to X_0, $g=0$, $deg=3$, through x_1, x_2:

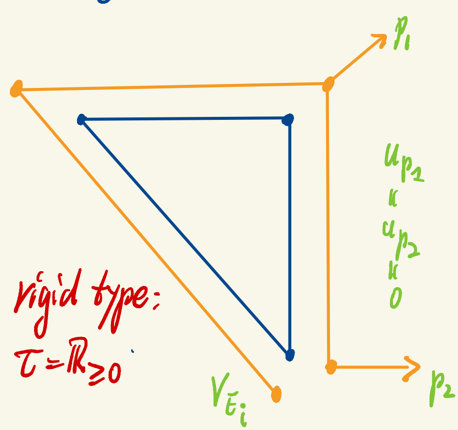
Two types:

I) $\forall i \; \exists! C$:

Log GW $= 9 \cdot 1$
$= 9$

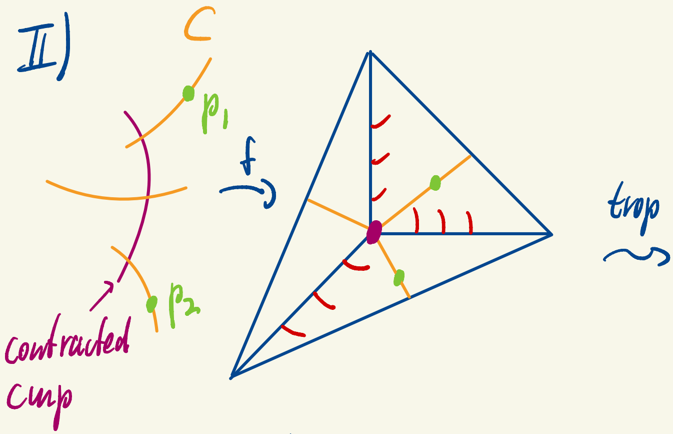


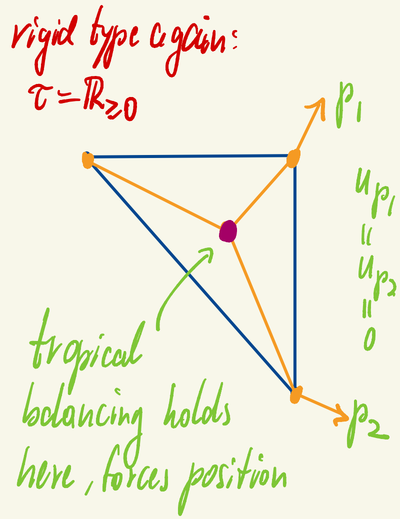
trop $\rightsquigarrow$



rigid type:
$\tau = \mathbb{R}_{\geq 0}$.

II)





A careful analysis shows that there are exactly 3 such stable log maps, with the same underlying stable map. These are all logarithmically unobstructed

$$\Rightarrow \quad \log GW = 3$$

Total count: $\boxed{12 = 9 + 3}$

Shows: Log GW theory of degenerations provide a natural refinement of the ordinary GW-theory of the general fibers.

The summands correspond to rigid tropical types!

i.e. $\dim \tau = 1$

Decomposition result: $[\mathcal{M}(X_0,\beta)] = \sum_{\tau} m_\tau \cdot [\mathcal{M}(X_0,\tau)]$

rigid type

$m_\tau \in \mathbb{N}$: lattice index of

$\tau_{\mathbb{Z}}^{gp} \longrightarrow (\Sigma_B = \mathbb{R}_{\geq 0})_{\mathbb{Z}}^{gp} = \mathbb{Z}$

Cor: $[\mathcal{M}(X_0,\beta)] = \sum_{\tau \text{ rigid}} \frac{m_\tau}{\mathrm{Aut}(\tau)} [\mathcal{M}(X_0,\tau)]$

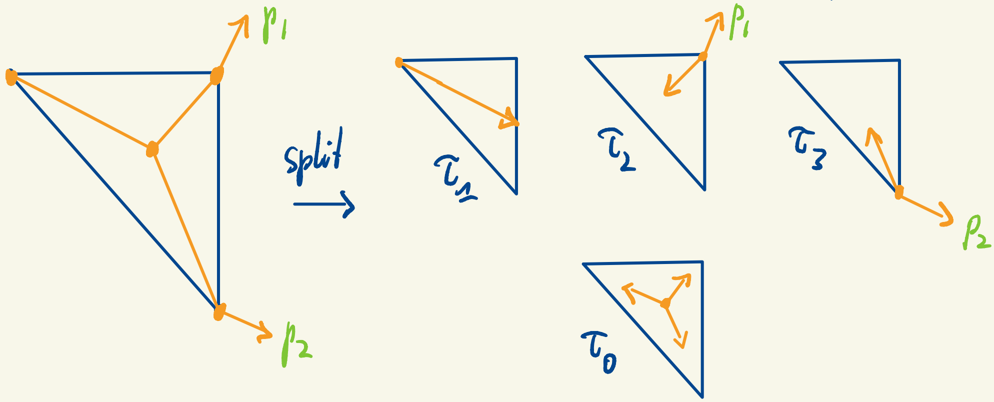
In expl: $12 = 9 + 3$

2. Splitting and gluing

Consider $X \to B$ log smooth, proper, $B = \begin{cases} pt \\ \text{std log point} \\ (\text{curve}, 0) \end{cases}$

Aim: Compute $[\mathcal{M}(X,\tau)]_{virt}$ by splitting τ along some edges [τ rigid or not]

Expl:



We will interpret τ_i as tropicalizations of punctured stable maps (whose contact orders may only lie in $\sigma(p)^{gp}$ rather than in $\sigma(p)$)
and prove:

Thm A

$$\begin{tikzcd} \mathcal{M}(X,\tau) \arrow[r] \arrow[d] & \prod_i \mathcal{M}(X,\tau_i) \arrow[d] \\ \mathfrak{M}^{ev}(X,\tau) \arrow[r, "\delta^{ev}"] \arrow[d] & \prod_i \mathfrak{M}^{ev}(X,\tau_i) \arrow[d] \\ \mathfrak{M}(X,\tau) \arrow[r, "\text{not cartesian}"] & \prod_i \mathfrak{M}(X,\tau_i) \end{tikzcd}$$

cartesian & compatible with vfc

finite, representable!

i.e. has non-stacky fibers

"Evaluation stacks": $\mathfrak{M}^{ev}(\mathcal{X},\tau) = \mathfrak{M}(\mathcal{X},\tau) \times_{\mathcal{X}^r} X^r$

for gluing along r edges [similarly for $\mathfrak{M}^{ev}(\mathcal{X},\tau_i)$]

Point: Working with $\mathfrak{M}^{ev}$ gets rid of the stacky nature of gluing stable punctured maps in $\mathfrak{M}(\mathcal{X},\tau_i)$.

Note: $X \longrightarrow \mathcal{X}$ smooth $\Rightarrow$ $\mathfrak{M}^{ev}(\mathcal{X}) \to \mathfrak{M}(\mathcal{X})$ smooth

$\Rightarrow$ Thm A reduces the computation of $[\mathcal{M}(X,\tau)]_{virt}$ to computing $\underbrace{\delta^{ev}_* [\mathfrak{M}^{ev}(\mathcal{X},\tau)]}$!

Defined in Chow theory & only depends on Σ and τ and the gluing strata in $\mathcal{X}$,

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Thm B: $\exists$ fs-cartesian diagram

$$\begin{tikzcd}
\widetilde{\mathfrak{M}}'^{ev}(\mathcal{X},\tau) \arrow[r, "\delta^{ev}"] \arrow[d, "ev"'] & \prod_i \widetilde{\mathfrak{M}}'^{ev}(\mathcal{X},\tau_i) \arrow[d] \\
\prod_{\substack{\text{splitting}\\ \text{edges}}} X \arrow[r, "\Delta"] & \prod_{\substack{\text{splitting}\\ \text{edges}}} X \times X
\end{tikzcd}$$

$\widetilde{\mathfrak{M}}^{ev} \longrightarrow \mathfrak{M}^{ev}$: enlarge log structure (from nodes & punctures)

$\mathfrak{M}'$ versus $\mathfrak{M}$: slightly weaker notion of marking

Point: $\widetilde{\mathfrak{M}}'^{ev} = \mathfrak{M}^{ev}$

$\Downarrow$

enough to compute $\delta^{ev}_*[\mathfrak{M}^{ev}]$

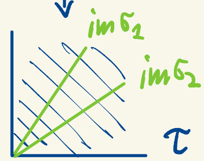
fs-cartesian: The fiber product in the category of fs log-schemes.

It is modeled on the fiber product of cones:

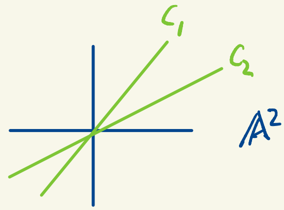
$$\begin{tikzcd} \sigma_1 \times_\tau \sigma_2 \arrow[r] \arrow[d] & \sigma_2 \arrow[d] \\ \sigma_1 \arrow[r] & \tau \end{tikzcd} \quad \rightsquigarrow \quad TV(\sigma_1) \times^{fs}_{TV(\tau)} TV(\sigma_2) = TV(\sigma_1 \times_\tau \sigma_2)$$

Expl:

$$\begin{tikzcd} \sigma_1 \times_\tau \sigma_2 = \{0\} \arrow[r] \arrow[d] & \sigma_2 = \mathbb{R}_{\geq 0} \arrow[d] \\ \sigma_1 = \mathbb{R}_{\geq 0} \arrow[r] & \tau \end{tikzcd}$$



shows:



$C_i \simeq (\mathbb{A}^1, 0)$ [$C_i \to \mathbb{A}^2$ not strict]

$\Rightarrow C_1 \times^{fs}_{\mathbb{A}^2} C_2 = \emptyset$!

Still, despite Thm B, $\delta_*^{ev}[\mathcal{M}^{ev}(X,\tau)]$ is often difficult to compute in practice.

Good case:

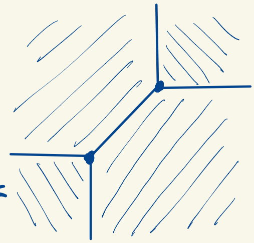
Yixian Wu: If the gluing strata are isomorphic to toric varieties [with their toric log str.] then $\delta_*^{ev}[\mathcal{M}^{ev}(X,\tau)]$ is a tropically computable sum of products of $[\mathcal{M}^{ev}(X,\omega_i)] \in A_*(\mathcal{M}^{ev}(X,\tau_i))$:

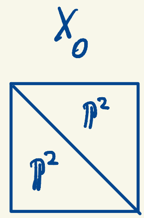
$$\boxed{\delta_*^{ev}[\mathcal{M}^{ev}(X,\tau)] = \sum_{\omega=(\omega_i)} (\text{mult's}) \cdot \prod_i [\mathcal{M}^{ev}(X,\omega_i)]}$$

The sum is over solutions (w_i) of a perturbed matching problem for the τ_i via a chosen displacement vector for the gluing edges.

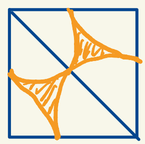
<u>Expl</u>: X = degeneration of $\mathbb{P}^1 \times \mathbb{P}^1$ to $\bigcup_2 \mathbb{P}^2 = TV(\Sigma_1)$,

$\Sigma_1 = \Sigma_X$ = cone over B =

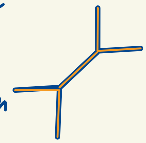




Curves of bidegree $(1,1)$ degenerate e.g. to

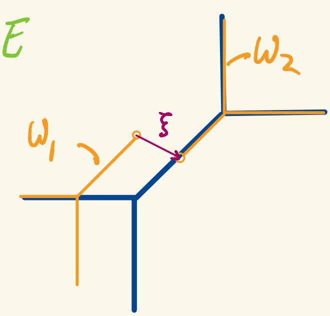


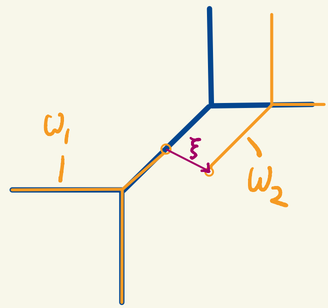
$\rightsquigarrow$ type τ $\cap$ 1-skeleton of B



Split at E, displace by ξ along E

$\rightsquigarrow$ two types $w = (w_1, w_2)$:





<u>Pf</u>: ξ defines a $\mathbb{G}_m$-action on $\mathcal{M}(\mathcal{X}, \tau)$

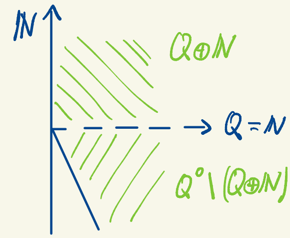
$\rightsquigarrow$ moves $\delta_*^{ev}[\overline{\mathcal{M}}^{ev}(\mathcal{X},\tau)]$ into lower-dimensional strata of $\prod_i \mathcal{M}^{ev}(\mathcal{X},\tau_i)$

3. Punctured curves and punctured GW theory

<u>Punctured curve:</u> Almost as before, but instead of $Q \oplus \mathbb{N}$ at a marked point admit submonoids $Q^\circ \subseteq Q \oplus \mathbb{Z}$ with

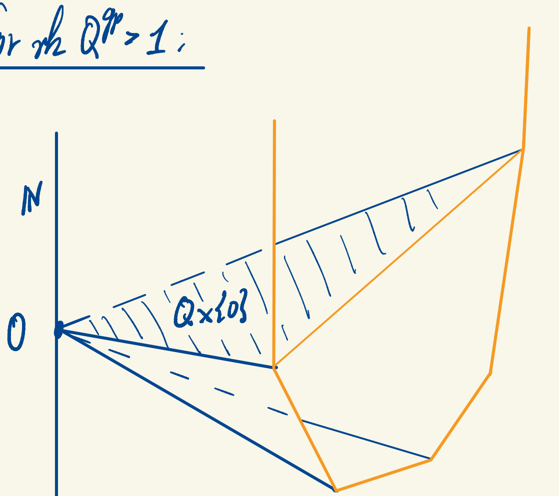
- $Q \oplus \mathbb{N} \subseteq Q^\circ$
- $\alpha(q,k) = 0$ whenever $k < 0$.

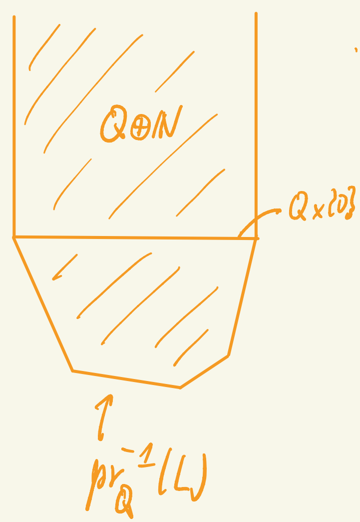
  $\alpha : \mathcal{M}_C \to \mathcal{O}_C$

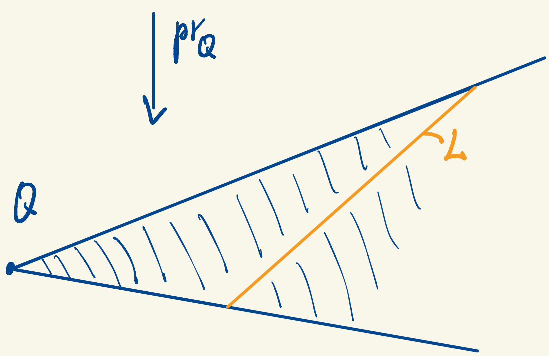


<u>Picture for $\mathrm{rk}\, Q^{gp} > 1$:</u>

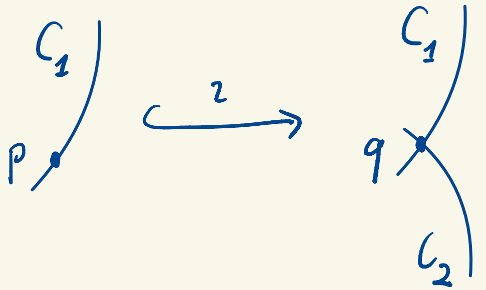
$Q \oplus \mathbb{Z}$







Punctures appear naturally when splitting nodes:



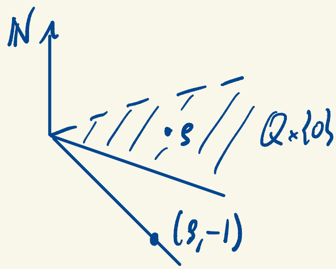
$C = C_1 \cup C_2$

$\mathcal{M}^\circ_{C_1} := \iota^* \mathcal{M}_C$

has a puncture at $p = \iota^{-1}(q)$

$\overline{\mathcal{M}}_{C,q} = Q \oplus_{\mathbb{N}} \mathbb{N}^2$

$s \leftarrow\!\shortmid\, 1 \longmapsto (1,1)$



$= \langle Q, (0,1), (s,-1) \rangle$

<u>Stable punctured maps</u>

$$\begin{array}{ccc} C^\circ & \xrightarrow{f} & X \\ \downarrow & & \\ X & & \end{array}$$

log smooth curve

s.th. $\overline{\mathcal{M}}_{C^\circ,p}$ is generated by

- $Q \oplus \mathbb{N}$
- $f^\flat(\overline{\mathcal{M}}_{X,f(p)})$ — may not be saturated

& $\mathrm{Aut}(\underline{C}^\circ, X)$ finite

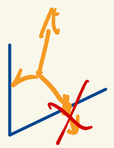
i.e. puncturing is as small as possible for f to exist.

Tropical interpretation: $\overline{\mathcal{M}}_{C^\circ,p} \neq Q \oplus \mathbb{N} \Rightarrow$ leg L_p is bounded, but extends to bdry of $\sigma(p)$.

<u>Yields</u>:
- tropical punctured maps
- types τ of " " "
- definition of basicness is unchanged $\rightsquigarrow \mathcal{M}(X,\tau)$, $\mathfrak{M}(X,\tau)$

• $\mathcal{M}(X,\tau) \longrightarrow \mathfrak{M}(X,\tau)$ virtually smooth

Important: $\mathfrak{M}(X,\tau)$ now may not be log smooth over $\mathbb{C}$, i.e. may not have toroidal singularities. Rather, it is locally isomorphic to the zero locus of an ideal in $\mathbb{C}[P_Z]$ generated by monomials, a possibly non-reduced union of orbit closures.



In particular, it may not be pure-dimensional, but it is if τ is "realizable" (puncturing legs point into σ_p))

Upshot: We have $[\mathcal{M}(X,\tau)]_{virt}$ in the realizable case.

Expl: $Bl_1(\mathbb{P}^1 \times \mathbb{P}^1)$

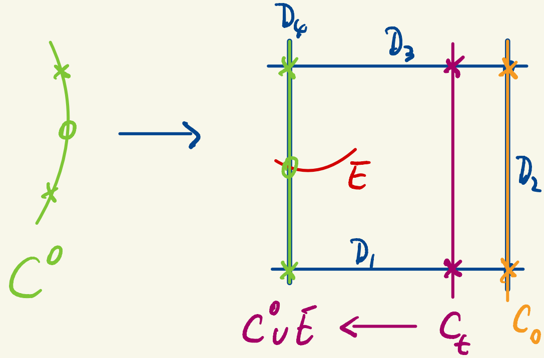


image of marked pt is free to move along D_4

tropically: