Inhomogeneous Gaudin algebras and cactus flower curves

$\mathfrak{g}$ semisimple Lie algebra, $n \in \mathbb{N}$

Goal: Find a basis for $V(\underline{\lambda}) = V(\lambda_1) \otimes \cdots \otimes V(\lambda_n)$

Plan: Find a large commutative subalg $A \subset (U\mathfrak{g})^{\otimes n}$ and hope that $V(\underline{\lambda})$ decomposes into distinct 1-dim submodules $\rightsquigarrow$ get a basis

(assume $\mathfrak{h} \subset A$, so we get a weight basis)

Given $\theta \in \mathfrak{h}^{reg}$, $\underline{z} \in \mathbb{C}^n \setminus \Delta$, we define inhomogeneous Gaudin Hamiltonian:

$i \in \{1, .., n\}$ $$H_i^\theta(\underline{z}) = \theta^{(i)} + \sum_{j \neq i} \frac{\Omega^{(ij)}}{z_i - z_j} \in (U\mathfrak{g})^{\otimes n}$$

added an "inhomogeneous term"

Theorem [Feigin–Frenkel–Rybnikov]

There exists a max commutative subalgebra $A_\theta(\underline{z}) \subset (U\mathfrak{g})^{\otimes n}$ containing $H_1^\theta(\underline{z}), \dots, H_n^\theta(\underline{z})$.

Fix θ, vary $\underline{z}$, $\quad \mathbb{C}^n \setminus \Delta / \mathbb{C} \longrightarrow$ subalgebras of $(U\mathfrak{g})^{\otimes n}$

translation

$$\underline{z} \longmapsto A_\theta(\underline{z})$$

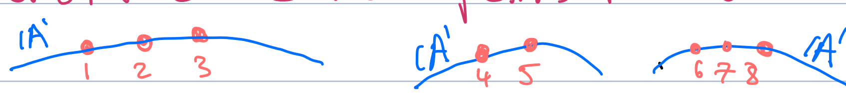
Question: What is the closure of the image of this map?

$\mathbb{C}^n/\mathbb{C}$ has a natural compactification called the matroid Schubert variety

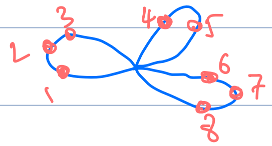
$$\mathbb{C}^n/\mathbb{C} = \{(\delta_{ij})_{i<j} : \delta_{ij} + \delta_{jk} = \delta_{ik}\} \subset \mathbb{C}^{\binom{n}{2}} \qquad \delta_{ij} = z_i - z_j$$

$$\overline{\mathbb{C}^n/\mathbb{C}} = \{(\delta_{ij})_{i<j} : \delta_{ij} + \delta_{jk} = \delta_{ik}\} \subset (\mathbb{P}^1)^{\binom{n}{2}}$$

allow distance between points to become infinite



$\delta_{14}=\infty$ $\delta_{15}=\infty$ $\delta_{47}=\infty$ etc.



eg $\overline{\mathbb{C}^3/\mathbb{C}} = \{(a,b,c)\in(\mathbb{P}^1)^3 : a+b=c\}$

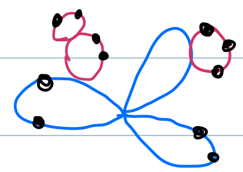
a singular variety!

In $\overline{\mathbb{C}^n/\mathbb{C}}$ we can still have equal marked points $\delta_{ij}=0$

Let $\overline{F}_n$ = iterated blowup of $\overline{\mathbb{C}^n/\mathbb{C}}$ along intersections of the loci $\{\delta_{ij}=0\}$

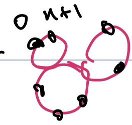
Points of $\overline{F}_n$ are "cactus flower curves"

blue components have smaller automorphism groups



In [Ilin-K-Li-Przytycki-Rybnikov] we defined $\overline{F}_n$ and constructed a deformation:

$$\begin{array}{ccccc} F_n & \subset & \mathcal{F}_n & \supset & \overline{M}_{n+2} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \in & \mathbb{C} & \ni & a\neq 0 \end{array}$$



So $\overline{F}_n$ comes from degeneration of $\overline{M}_{n+2}$ as two marked pts come together.

Theorem [IKR]

① The map $\mathbb{C}^n\setminus\Delta/\mathbb{C} \longrightarrow$ subalg of $(U\mathfrak{g})^{\otimes n}$

$\underline{z} \longmapsto A_\theta(\underline{z})$

compactifies to $\overline{F}_n \longrightarrow$ subalg of $(U\mathfrak{g})^{\otimes n}$

In fact we get $\mathcal{F}_n \longrightarrow$ subalg of $(U\mathfrak{g})^{\otimes n}$

if $a\neq 0$, we a trigonometric Gaudin alg $A^{trig}_{a^{-1}\theta}(\underline{z})$

$QH^0_{\mathbb{G},1}(\overline{W}^{\lambda}_{\mu})$

② For any $\underline{z} \in \overline{F}_n(\mathbb{R})$, the algebra $A_\theta(\underline{z})$ (or $A^{\text{trig}}_{\alpha,\theta}(\underline{z})$) acts semisimply on $V(\underline{\lambda})_\mu$ with simple spectrum.

(for $a \neq 0$ this is the $QH^\bullet_{\theta,1}(W^{\lambda}_{\mu})$)

We can define a cover $\mathcal{E} \ni \{ L \subset V(\lambda)_\mu : L$ is an eigenline for action of $A_\theta(\underline{z})$ $\}$

$$\begin{array}{ccc} \mathcal{E} & \ni & \{\ldots\} \\ \downarrow & & \downarrow \\ \overline{F}_n(\mathbb{R}) & \ni & \underline{z} \end{array}$$

finite set of size $\dim V(\lambda)_\mu$

Question

What is the monodromy of this cover? $\pi_1(\overline{F}_n(\mathbb{R})) \curvearrowright$ fibre of $\mathcal{E}$

In [IKLPR], we studied the topology of

$$\begin{array}{ccccc} \overline{F}_n(\mathbb{R}) & \subset & \overline{\mathcal{F}}_n(\mathbb{R}) & \supset & \overline{M}^\sigma_{n+2}(\mathbb{R}) \\ \downarrow & & \downarrow & & \downarrow \\ 0 \in & & \mathbb{R} & \ni & a \neq 0 \end{array}$$

Theorem

We have $\pi_1^{S_n}(\overline{M}^\sigma_{n+2}(\mathbb{R})) \xrightarrow{\text{inclusion of general fibre}} \pi_1^{S_n}(\overline{\mathcal{F}}_n(\mathbb{R})) \xleftarrow[\sim]{\text{inclusion of special fibre}} \pi_1^{S_n}(\overline{F}_n(\mathbb{R}))$

$\pi_1^{S_n}(\overline{M}^\sigma_{n+2}(\mathbb{R})) \cong AC_n$ affine Cactus group

$\pi_1^{S_n}(\overline{F}_n(\mathbb{R})) \cong vC_n$ virtual cactus group

- the real structure on $\overline{M}^\sigma_{n+2}(\mathbb{R})$ is not the standard one:
  $\{ (C, z_0, \ldots, z_{n+1}) : \overline{z}_0 = z_{n+1} \quad z_i \in C(\mathbb{R}),\ i \neq 0, n+1 \}$
- we add S_n-equivariance to make the group simpler.

Cor

The monodromy action of $\pi_1^{S_n}(\overline{M}^\sigma_{n+2}(\mathbb{R}))$ on the fibres of the cover $\mathcal{E}$ factors through the map $AC_n \to vC_n$

(monodromy of eigenvectors for quantum multiplication)

Def Cactus group and its variants $C_n \to S_n$

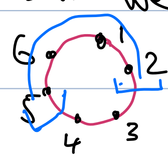
$s_{ij} \mapsto$ (permutation reversing $i, \ldots, j$)

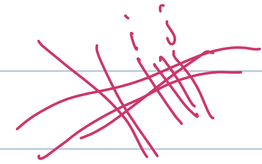
$C_n = \langle s_{ij}$ for $1 \leq i < j \leq n \mid s_{ij}^2 = 1,$

$s_{ij}s_{kl} = s_{kl}s_{ij}$ if $[k,l]\cap[i,j]=\emptyset$, $s_{ij}s_{kl} = s_{i+j-l\; i+j-k}\, s_{ij}$ if $[k,l]\subset[i,j]$)

AC_n = same as C_n but we allow intervals on a circle

affine cactus

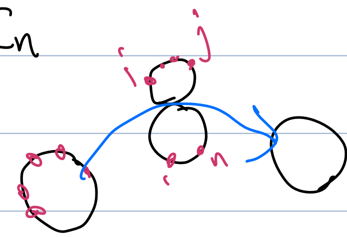




$vC_n = C_n * S_n \,/\, w s_{ij} = s_{w(i)\, w(j)}\, w$ if $w(i+k) = w(i)+k$ for $k=1,\dots,j-i$

virtual cactus

There are inclusions $C_n \hookrightarrow AC_n \hookrightarrow vC_n$



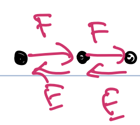
Theorem [Davis-Januszkiewicz-Scott]

$$C_n = \pi_1^{S_n}(\overline{M}_{n+1}(\mathbb{R}))$$

Recall our goal was to determine the monodromy action of $\pi_1(\overline{M}^{o}_{n+2}(\mathbb{R}))$ on the fibres of the cover $\mathcal{E}$

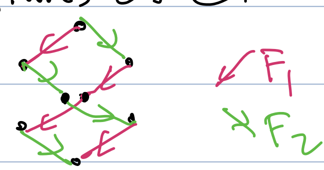
By Theorem above, it suffices to study the action of $vC_n = \pi_1^{S_n}(\overline{F}_n(\mathbb{R}))$ on the fibres of $\mathcal{E}$.

$V(2)$



$V(2,1,0)$

adjoint rep of $\mathfrak{sl}_3$



Crystals

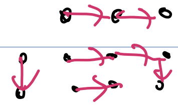
To every representation of a semisimple Lie algebra, there is a combinatorial object called a crystal

a graph: vertices = basis of V

edges = actions of E_i, F_i on the basis

$V(\lambda_1)\otimes\cdots\otimes V(\lambda_n) \rightsquigarrow B(\lambda_1)\otimes\cdots\otimes B(\lambda_n)$

$V(2)\otimes V(1)$



underlying set is $B(\lambda_1)\times\cdots\times B(\lambda_n)$

but "non-symmetric" crystal structure

Theorem [Henriques–K 2004]

There is an action of C_n on $B(\lambda_1)\otimes\cdots\otimes B(\lambda_n)$

(the category of crystals is a "coboundary monoidal category")

Also we can act by S_n on $B(\lambda_1)\times\cdots\times B(\lambda_n)$ by naive permutations.

Theorem [K–Rybnikov to appear]

These combine to give an action of vC_n on $B(\lambda_1)\times\cdots\times B(\lambda_n)$

There is a vC_n-equivariant bijection

$$B(\lambda_1)\times\cdots\times B(\lambda_n) = \text{eigenlines for } A_\theta(\underline{z}) \text{ acting on } V(\lambda_1)\otimes\cdots\otimes V(\lambda_n).$$

Corollary

The monodromy action of $\pi_1(\overline{M}^{o}_{n+2}(\mathbb{R})) = AC_n$ on the eigenvectors for the quantum cohomology of W^λ_μ is given by $AC_n \to vC_n \circlearrowright B(\lambda_1)\times\cdots\times B(\lambda_n)_\mu$.

Conj

$A^{trig}_\theta(\underline{z})$ acting on $V(\lambda)_\mu = H^\circ_{\theta,t}(W^\lambda_\mu)$.

$\underline{z} =$