5) Holomorphic Floer theory and DT invariants [B, 2210.17001]

Last time:

CY3   Quiver DT $\longleftrightarrow$ Log GW cluster varieties

$\mathcal{E}$

$\downarrow$ $\downarrow$

Categorification   ?

COHA ...

$\dim = \mathrm{rk}\, K_0(\mathcal{E})$

$(\mathbb{C}^*)^{\mathrm{rk}\, K_0(\mathcal{E})}$

$\{\ ,\ \}$

Holomorphic symplectic

DT wall-crossing (4d)

$X \times \mathbb{R}^4$
CY3

vs

Picard-Lefschetz theory / $\mathbb{C}$-Morse theory
2d Cecotti-Vafa wall-crossing
QH^*(Fano)

Known analogy (Joyce, Bridgeland, ...)

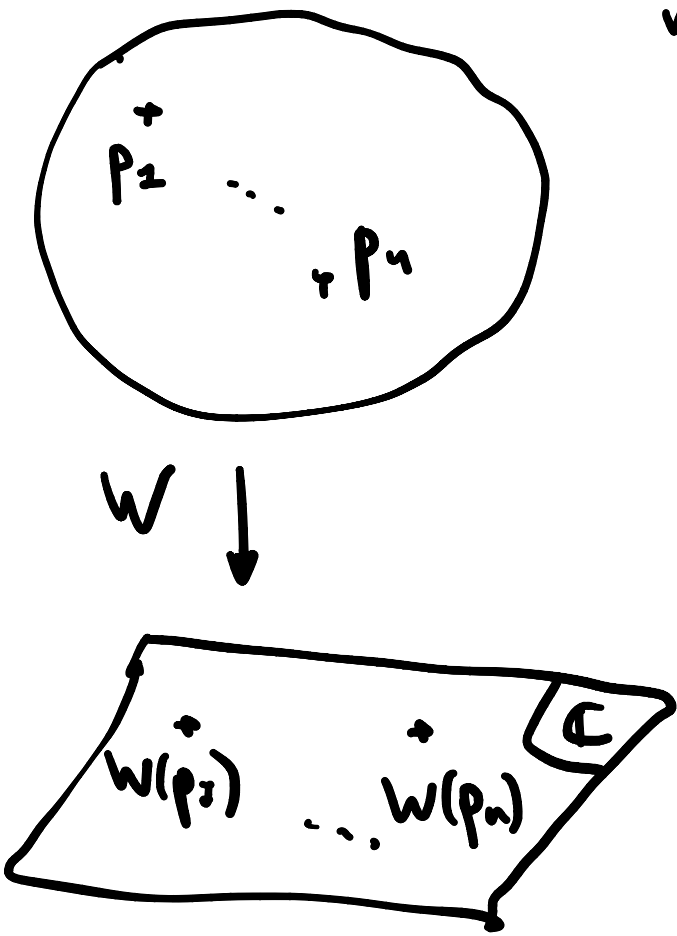
Q] Is it more than an analogy?

2d Wall-crossing / $\mathbb{C}$-Morse theory

P Kähler manifold $\quad W : P \to \mathbb{C}$ holomorphic function

(P, W) (e.g. mirror of a Fano variety)

Assume finitely many critical points $p_1, \ldots, p_n \in P$
non-degenerate



$\mathbb{C}$ - Morse theory ?

$\mathbb{R}$ - Morse theory (P, $f: P \to \mathbb{R}$ C^∞)

C^∞ manifold

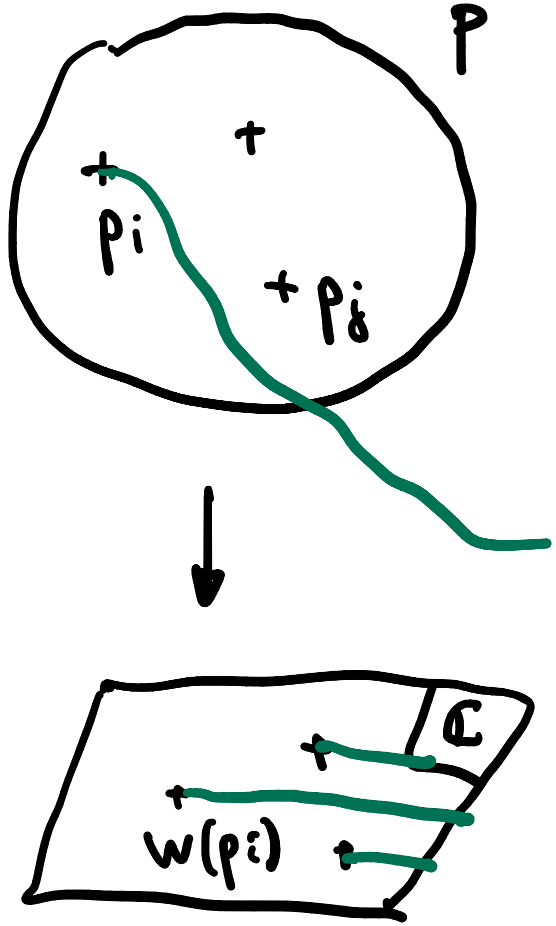
$\rightsquigarrow$ Morse complex $\rightsquigarrow$ Morse homology

$H^*(P, f)$

$\mathbb{C}$ - Morse theory (P, $W: P \to \mathbb{C}$ holom)

Kähler

$\longrightarrow$ Fukaya category $\mathrm{Fuk}(P, W)$

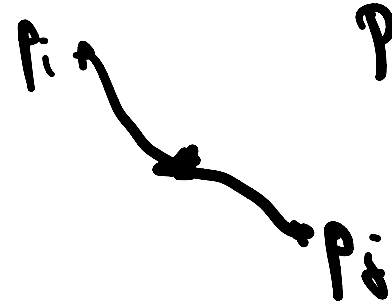


W holomorphic

$\mathrm{Re}\left(\frac{W}{\zeta}\right)$ $|\zeta| = 1$ real Morse function

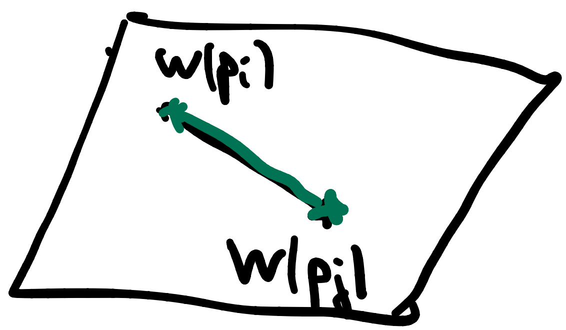
Critical points p_i: All same index

$\Rightarrow$ No gradient flow lines for generic ζ



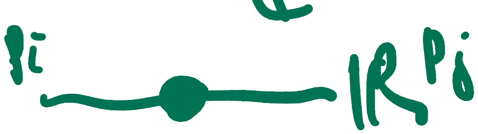
Possible for $\zeta = \zeta_{ij}$

$$\zeta_{ij} = \frac{W(p_i) - W(p_j)}{|W(p_i) - W(p_j)|}$$



$\mu_{ij} = \#\ \zeta_{ij}$-gradient flow lines between p_i and p_j

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r] & (P, W) \arrow[d] \\ & \mathbb{C} \end{tikzcd}$$



[$= \#\ \zeta_{ij}$-solitons $= \#$ 2d BPS states]

Categorification BPS_{ij} complex p_i

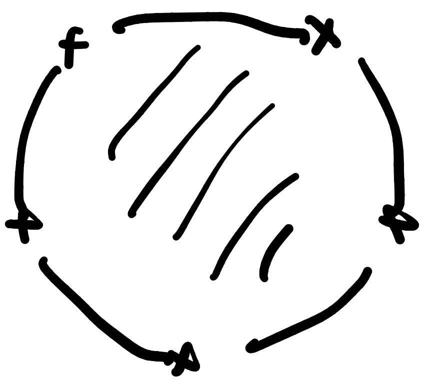
p_j

solts of ζ- instanton eq

Perturbed J-holom curves

by $\mathrm{Im}\left(\frac{W}{\zeta}\right)$

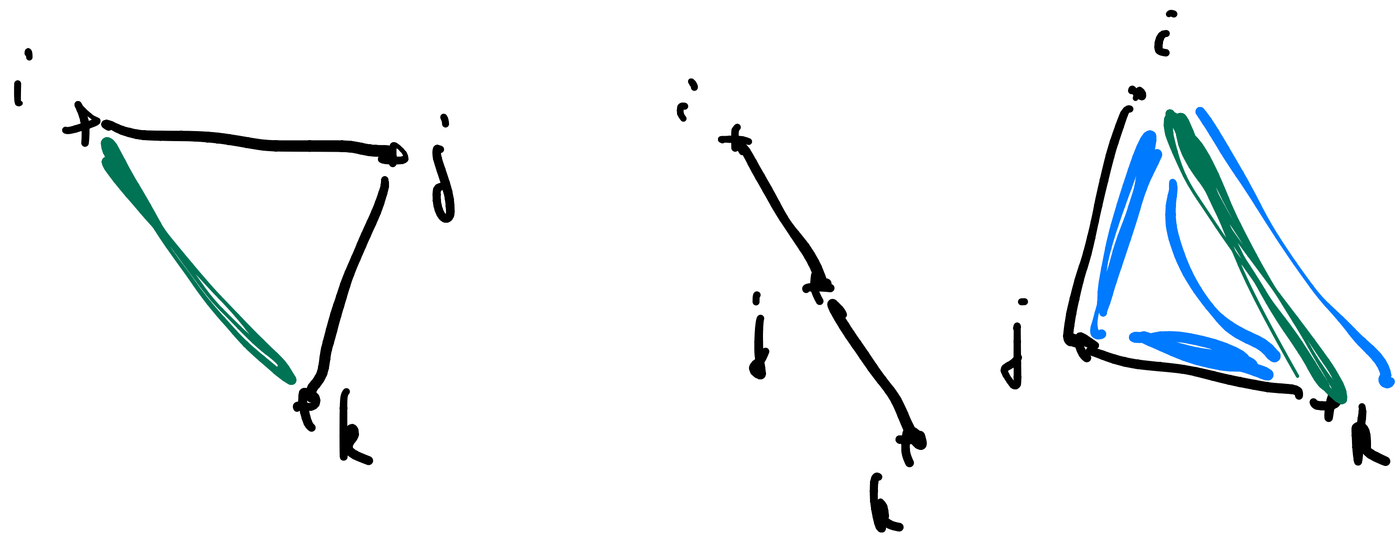
BPS_{ij} + Higher operations $\longrightarrow$ Fukaya category $\mathrm{Fuk}(P, W)$



Conj: [Haydys, Gaiotto-Moore-Witten]

Vary W $\longrightarrow$ Wall-crossing formula

W $\llcorner\mathbb{C}$



$\mu_{ik} \longrightarrow \mu_{ik} + \mu_{ij}\,\mu_{jk}$

2d Wall-Crossing formula
(Cecotti-Vafa)
(Picard-Lefschetz)

(μ_{ij}) $\quad (\qquad)$

Ω_γ

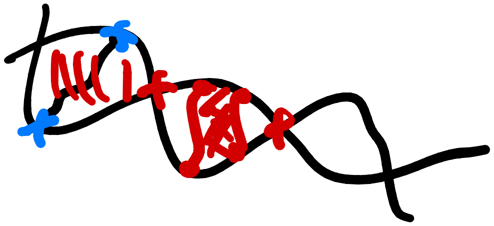
Analogy:

2d	4d
(P, W)	CY_3
$\{p_i, p_j\}$	$\gamma \in \Gamma$
$W(p_i) - W(p_j) \in \mathbb{C}$	$Z_\gamma \in \mathbb{C}$
μ_{ij}	Ω_γ
BPS_{ij}	Categorical Ω_γ

Q: $\exists$? (P, W) reproducing DT CY_3 story?

Floer theory

(M,ω) $\mathbb{R}$-symplectic manifold



L_0

L_1

Lagrangian P = space of paths $[0,1] \to M$ ending on L_0 & L_1

Action functional $\tilde{P} \to \mathbb{R}$

$\int \omega$

$\mathbb{R}$-Morse theory

$\to$ Floer complex $\to HF^*(L_0, L_1)$

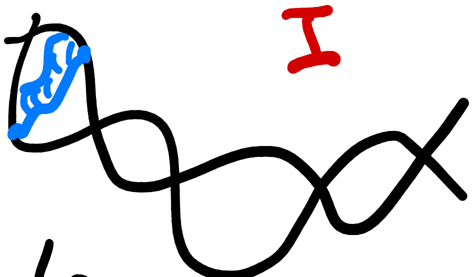
$\forall L_i \to \mathrm{Fuk}(M,\omega)$

Holomorphic Floer theory [Kontsevich-Soibelman]

(M, Ω_I) $\mathbb{C}$-symplectic P

$\tilde{P} \to \mathbb{C}$

$\int \Omega$



I

L_0

L_1 Ω_I-holom Lag

$\to \mu_{ij}$

$\to BPS_{ij}$ $\forall i,j \in L_0 \cap L_1$

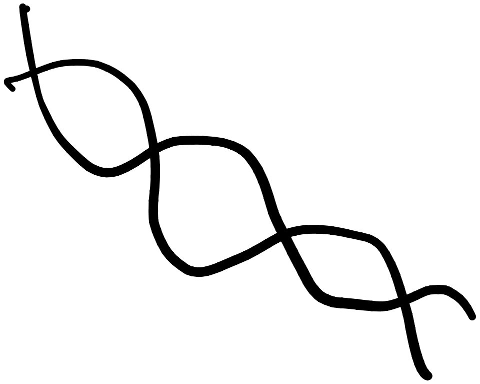
$\to \mathrm{Fuk}(P,W) = \mathrm{Hom}(L_0, L_1)$ category

$\forall L_i \to \mathrm{Fuet}(M, \Omega_I)$ Fueter 2-category

$\omega_\zeta = \mathrm{Re}\left[\frac{\Omega_I}{\zeta}\right]$

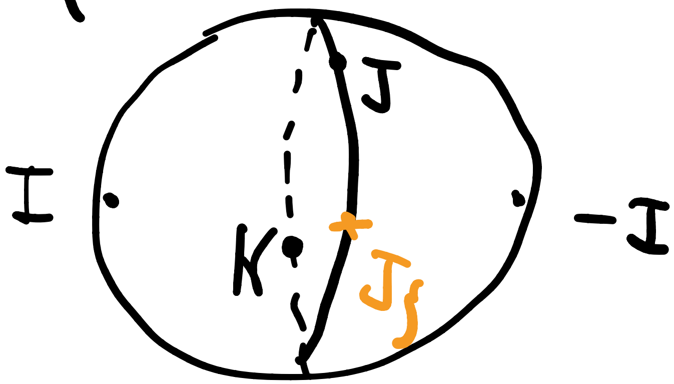
$\mathrm{Fuk}(\omega_\zeta)$

$I \quad J \quad K \qquad I^2 = J^2 = K^2 = IJK = -1$



Hyperkähler Triple

$\zeta = e^{i\theta} \qquad J_\zeta = (\cos\theta) J + (\sin\theta) K$



Twistor sphere

$aI + bJ + cK$

$a^2 + b^2 + c^2 = 1$

ζ-soliton $\leftrightarrow$ J_ζ-holom curves

ζ-instanton $\leftrightarrow$ $\mathbb{R}^2 \times [0,1] \to M$

Fueter equation

$$I \frac{\partial u}{\partial t} + J \frac{\partial u}{\partial r} + K \frac{\partial u}{\partial s} = 0$$

[B, 2022] [Doan-Rezchikov, 2022]

$$(M, \Omega_I) \xrightarrow[?]{} \mathrm{Fuct}(M, \Omega_I)$$

2-category (A-model)

∂-conditions of 3d σ-model $\mathcal{N}=4$

$\mathbb{R}^3 \to (M, \Omega_I)$

$\left(\neq RW(M, \Omega_I) \right.$ 2-category (B-model)$\left.\right)$

$\mathbb{R}^2 \times [\ \ ]$

$\simeq$ $\quad L_0 \quad L_1$

Expected to play a role in 3d mirror symmetry

(Gammage-Hilburn Gaiotto ...)

DT CY3 $\quad \mathbb{R}^4 \times X \quad \mathcal{E}$ CY3

$$\begin{array}{cc} \mathbb{R}^4 & X \\ \uparrow & \uparrow \\ \text{4d } \mathcal{N}=2 & \text{CY3} \end{array}$$

GADE

G-Higgs on C

$$X \quad \text{ADE}$$
$$\downarrow$$
$$C$$

$\mathbb{R}^3 \times S^1$ 3d σ-model $\to \mathcal{M}$

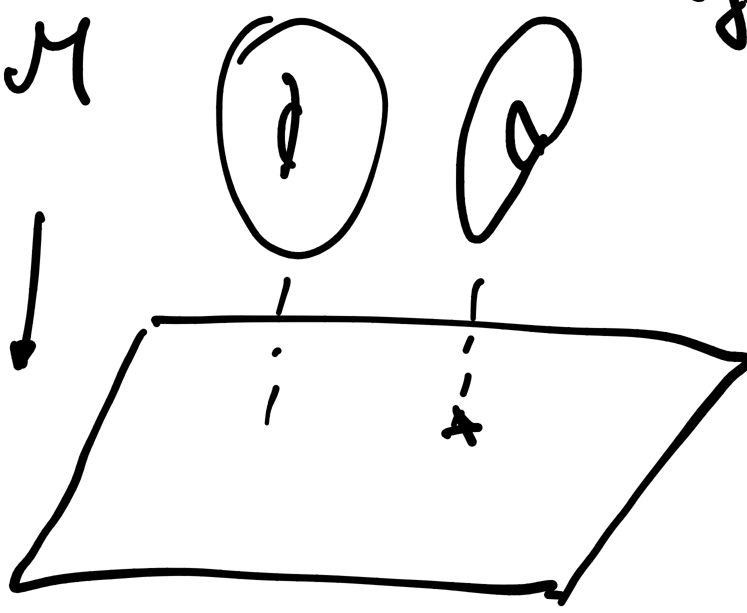
Coulomb branch of theory on $\mathbb{R}^3 \times S^1$

Seiberg-Witten integrable systems

$$\mathcal{M} \quad \Omega_I$$
$$\downarrow$$
$$B \to \mathrm{Stab}(\mathcal{E})$$

$B \leftarrow$ 4d Coulomb branch

$$\begin{array}{c} \mathcal{M} \\ \downarrow \\ B \end{array}$$



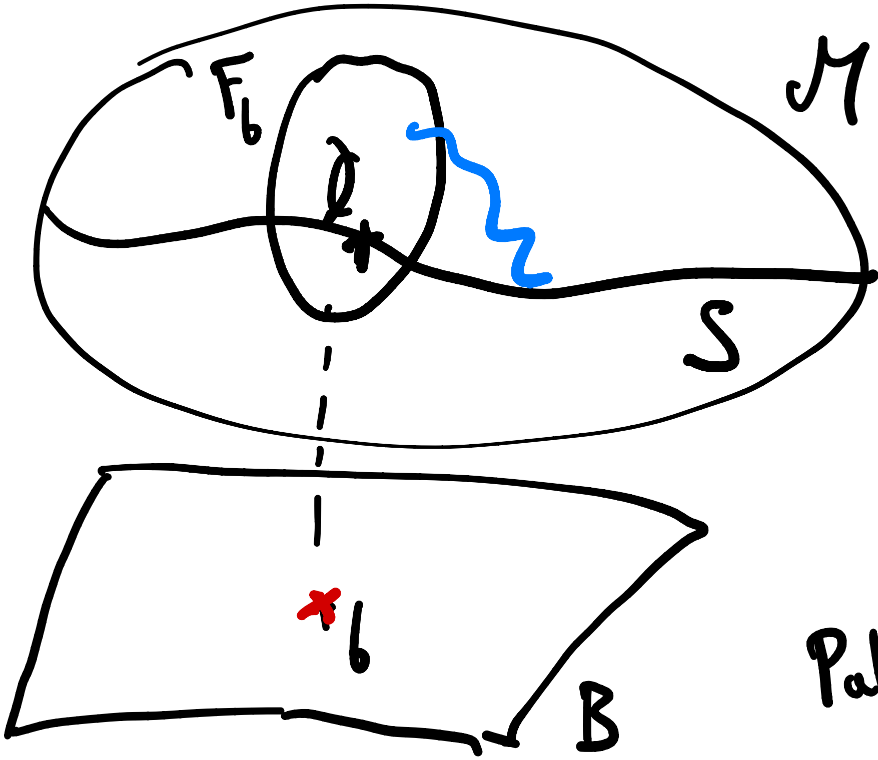
B

IIB/X $\quad$ DT $\mathrm{Fuk}(X) = \mathcal{E}$

$B \sim \mathbb{C}$-moduli of $X \quad \dim_{\mathbb{C}} = b_3(X)$

$\mathcal{M} \sim$ family of intermediate Jacobians

$$H^3(X,\mathbb{R})/H^3(X,\mathbb{Z})$$



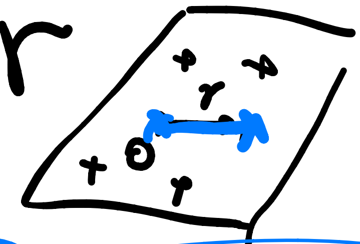
$$\mathcal{M} \to B$$

S section

F_b fiber over B

$H_1(F_b) \sim H_3(X,\mathbb{Z}) \sim \Gamma$

$\mathrm{Paths}(F_b, S) \quad \pi_1 \simeq \Gamma$



Conj: (B, 2022)

For non-compact CY3

4d $\mathcal{N}=2$ QFT

(no gravity)

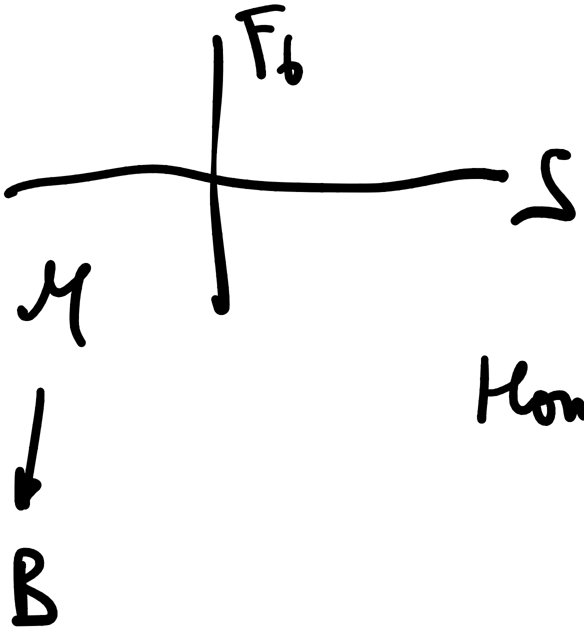
$\Omega^b_\gamma \leftrightarrow \chi(o,\gamma)$ J_ξ-holom curves

Categorical DT $\leftrightarrow$ $\mathrm{BPS}(o,\gamma)$

$(\mathcal{M}, J)$

~ Cluster Variety

$$\to \Omega = \log GW(\mathcal{M}, J)$$



Expectations:

$\mathrm{Hom}_{\mathrm{Fuk}}(S,S) = \otimes$-categorification of cluster algebras / Θ-functions

$$\mathrm{Hom}_{\mathrm{Fuk}}(S,S) \longrightarrow \mathrm{Hom}_{\mathrm{Fuk}}(S,F_b)$$

$=$ categorification of expansion of Θ functions on cluster charts.