0. Mirror Symmetry

Physics duality of IIA/IIB string ...

Math

	A-side (symplectic geometry)	B-side (complex geometry)
enumerative:	Gromov-Witten invts	variation of Hodge structures (period integrals)
homological:	Fukaya category	$D^b\mathcal{O}_X$

Basic question: For which pairs of spaces?

Ad hoc/synthetic:
- Batyrev-Borisov (CICY $\subseteq$ toric)
- Berglund-Hübsch
- toric degenerations (Gross/S. 2002-2012)

Dream result: $(Y \text{ A-geom}) \longrightarrow (X \text{ B-geom})$

for a natural class of Y's

Intrinsic mirrors:

$$\boxed{\begin{array}{l}(Y,D) \text{ nc (toroidal)},\\ \text{weak (log-) Calabi-Yau}\end{array}} \Longrightarrow \boxed{\begin{array}{c}\mathcal{X}\\ \downarrow \text{ - affine}\\ \operatorname{Spec} \mathbb{C}[NE(Y)]\end{array}}$$

absolute case

$K_Y + D = \sum a_i D_i$, $a_i \geq 0$

[e.g. Y Fano, $D \sim K_Y$]

[exists for any maximal degeneration of CY-mflds!]

Variant:

$(\mathcal{Y}, \mathcal{D})$ toroidal

$$\boxed{\begin{array}{ccccc} Y_0 & \subseteq & \mathcal{D} & \subseteq & \mathcal{Y}\\ \downarrow & & & & \downarrow\\ 0 & & \in & & C\end{array}} \Longrightarrow \boxed{\begin{array}{c}\mathcal{X}\\ \downarrow \text{ - projective}\\ \operatorname{Spec} \mathbb{C}[NE(\mathcal{Y})]\end{array}}$$

affine curve

relative case

<u>Rem:</u> $D / \mathcal{D}$ should have a 0d stratum to produce a mirror of the same dimension.

[Spec in Fano case]

Answer to basic question:
$$\boxed{\mathcal{X} = \begin{cases} \operatorname{Spf} QH^0(Y,D) & \text{absolute}\\ \operatorname{Proj} QH^0(\mathcal{Y},\mathcal{D}) & \text{relative}\end{cases}}$$

I'll explain the construction by going through one example.

I. A toroidal degeneration of $(\mathbb{P}^2, E)$

- Givental / Hori-Vafa : mirror of $Y = \mathbb{P}^2 \setminus D$, $D = \bigcup_3 \mathbb{P}^1$: $X = (\mathbb{C}^*)^2$ of pair (Y,D) : add "superpotential" $W = x + y + \frac{t}{xy}$

$$t = e^{-\int_\ell \omega} \curvearrowright \text{Kähler class}$$

- Intrinsic mirror of (Y,D): $\mathcal{X} = (xyz = t^\ell) \subseteq \mathbb{A}^3 \times \mathbb{A}^1_t$

$$\overset{\uparrow}{\mathbb{C}[NE(Y)]}.$$

More interesting: $D = E \subseteq \mathbb{P}^2$ smooth cubic.

- Intrinsic mirror of $(Y,D) = (\mathbb{P}^2, E)$: $\mathcal{X} \simeq \mathbb{A}^1_{\mathbb{C}[[t]]}$

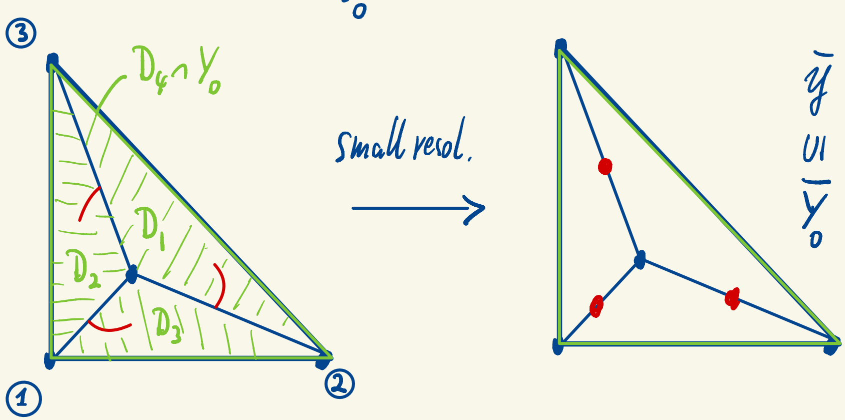
While the structure coefficients of $\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$ are interesting ($\to$ Yu Wang's thesis), the geometry has the wrong dimension and is boring.

- Instead, look at toroidal degeneration $\begin{array}{c}(\mathcal{Y},\mathcal{D})\\ \downarrow \\ C\end{array}$ of $(\mathbb{P}^2, E)$:

$$\begin{tikzcd} \mathcal{Y} \arrow[dr, "\kappa"] \arrow[dd, "\pi"'] & \\ & \bar{\mathcal{Y}} = V\big(X_0X_1X_2 - s(X_3 - \overset{\text{deg}=3,\ \text{general}}{f}(X_0,X_1,X_2))\big) \subseteq \mathbb{P}^3(\underset{X_0}{1},\underset{X_1}{1},\underset{X_2}{1},\underset{X_3}{3}) \arrow[dl] \\ C = \mathbb{A}^1_s & \end{tikzcd}$$

$$\mathcal{D} = \underbrace{D_1 \cup D_2 \cup D_3}_{Y_0} \cup D_4$$

κ: $\mathcal{Y} \supseteq Y_0$



$\bar{\mathcal{Y}} \supseteq \bar{Y}_0$

Intrinsic mirror: [we took $NE(\mathcal{Y}) \to \mathbb{N}$, $C \mapsto C \cdot D_4$]

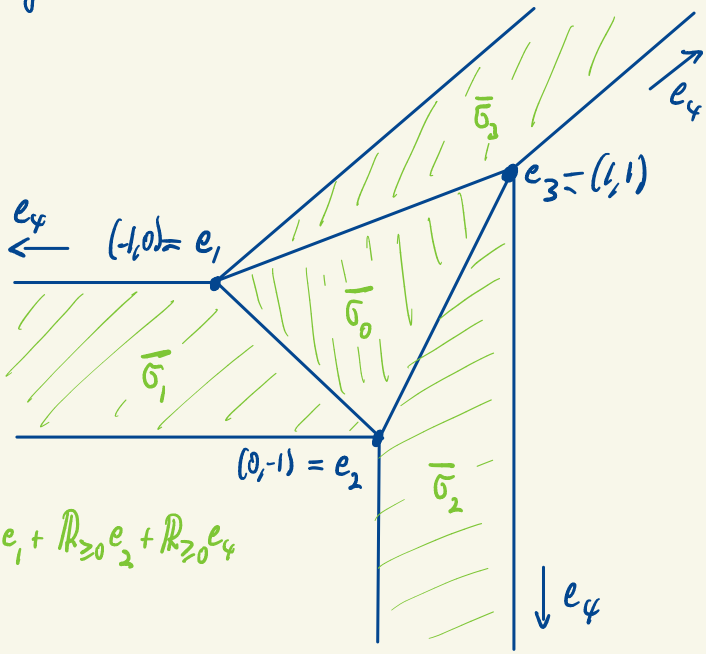
<u>Thm</u> $$QH^0(\mathcal{Y},\mathcal{D}) = \mathbb{C}[\![t]\!][\underbrace{X,Y,Z}_{\text{deg}=1},U,\underset{\text{deg}=0}{W}] / (XYZ - U^3,\ WU - t\cdot(X+Y+Z))$$

<u>Note</u>: Dehomogenizing at U yields the Hori-Vafa mirror and $W = t\cdot(x+y+z)$.

II. Step 1: Tropicalize $(\mathcal{Y}, \mathcal{D})$

$g = \mathrm{Trop}(\pi) : \Sigma_{\mathcal{Y}} \longrightarrow \Sigma_{\mathbb{A}^1} = \mathbb{R}_{\geq 0}$

$g^{-1}(1)$:



$\sigma_1 = \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0} e_4$

etc.

<u>Rem:</u> If $K_{\mathcal{Y}} + \mathcal{D} = \sum_i a_i \mathcal{D}_i$ with some $a_i > 0$, one needs to further restrict to the KS-skeleton $\subseteq \Sigma_{\mathcal{Y}}$.

III. Intrinsic mirror ring via structure coefficients

as a module: (⇒ flat)

$$QH^0(Y,D) := \bigoplus_{p\in\Sigma_Y(Z)} \mathbb{C}[\![NE(Y)]\!]\cdot \vartheta_p$$

ring structure:

$$\vartheta_p\cdot\vartheta_q = \sum_{\substack{r\in\Sigma_Y(Z)\\ A\in NE(Y)}} N^A_{pqr}\cdot \vartheta_r\cdot t^A$$

N^A_{pqr} : punctured GW-inv with contact order p, q and $-r$.

Note: $\Sigma_Y(Z) = \bigcup_{\sigma\in\Sigma_Y} \sigma_{\mathbb{Z}}$ is the set of non-negative contact orders.

[contacts with D]

Thus N^A_{pqr} requires punctures unless $r=0$.

Thm (Gross/S. '19) $QH^0(Y,D)$ is associative.

For better control: Alternative construction via wall structures.

Makes also contact to our previous synthetic mirror construction (2002–2012), which also retrieves the Batyrev–Borisov case of CICY ⊆ toric.

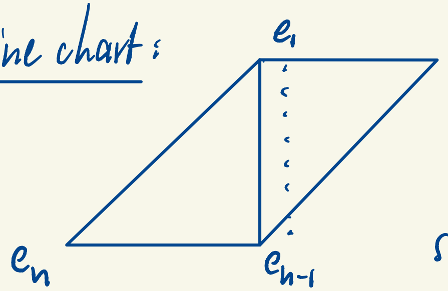
IV. Step 2: Affine structure on $\Sigma(\mathcal{Y}) \setminus \mathrm{codim}\, 2$

nc case: (toroidal: desingularize torically first)

adjunction $\Rightarrow D_{i_1} \cap \dots \cap D_{i_r} \subseteq \mathcal{D}$ also log-CY

$\Rightarrow$ any 1-stratum $\simeq (\mathbb{P}^1, 2\text{pts})$ [now essential: $\mathcal{Y}$ has 0d strata]

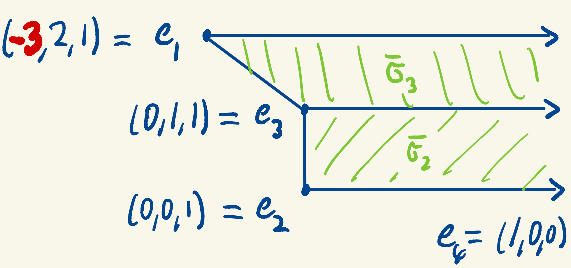
affine chart:



$e_1, \dots, e_{n-1}$ span $(n-1)$-simplex $\rightsquigarrow$ curve $C \simeq \mathbb{P}^1 \subseteq \mathcal{D}$

s.th.
$$\boxed{e_n + e_n' = -\sum_{\mu=1}^{n-1} (D_\mu \cdot C) \cdot e_\mu}$$

Expl: $C = D_2 \cap D_4$ in our degeneration $(\mathcal{Y}, \mathcal{D})$ of $(\mathbb{P}^2, E)$:



$$e_1 + e_2 = -(D_3 \cdot C)\cdot e_3 - (D_4 \cdot C)\cdot e_4$$
$$-D_3 \cdot C = (D_1 + D_2)\cdot C = 2$$
$$D_4 \cdot C = \frac{1}{3} \cdot E \cdot E = 3$$

$$\Rightarrow \quad e_1 = -(0,0,1) + 2\cdot(0,1,1) - 3\cdot(1,0,0) = (-3,2,1)$$

V. Step 3: Wall structure

$\mathbb{Z}$-affine charts provide Laurent polynomial rings:

$$\sigma \in \Sigma(Y) \rightsquigarrow \mathbb{C}[\sigma_{\mathbb{Z}}^{gp}] \simeq \mathbb{C}[x_1^{\pm},\dots,x_n^{\pm}].$$

<u>wall:</u> a $\mathrm{codim}_{\mathbb{R}}=1$ $\mathbb{Q}$-polyhedral cone $\rho \subseteq \sigma$

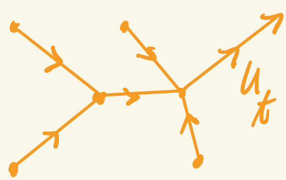
& $f_\rho \in \mathbb{C}[\underset{\substack{\| \\ NE(Y)}}{P}][\rho_{\mathbb{Z}}^{gp}] \simeq \mathbb{C}[P][x_1^{\pm},\dots,x_{n-1}^{\pm}]$

<u>wall structure:</u> a union of walls.

$(Y,D) \rightsquigarrow$ canonical wall structure on $\Sigma(Y)$:

one wall ρ for each <u>wall type t</u> = type of tropical punctured map with one puncture with endpoint varying in $(n-1)$-dim'l family.

$$\boxed{f_\rho = \exp\left(k_t \cdot W_{t,A}\, t^A z^{-u_t}\right)}$$

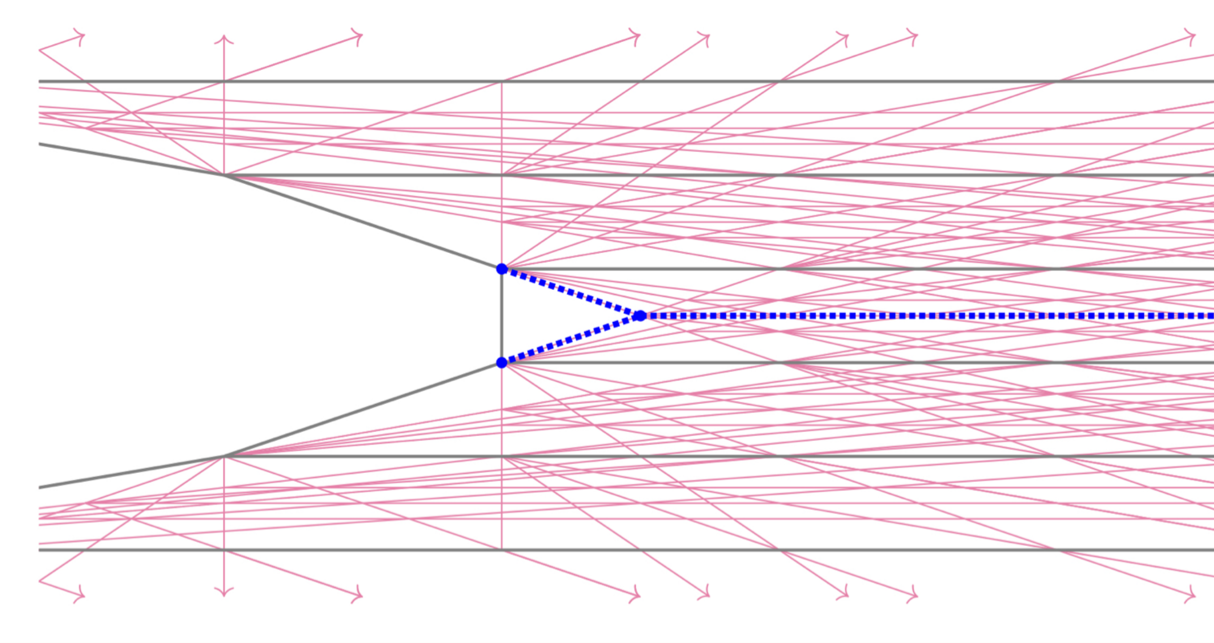


u_t: contact order at puncture

$W_{t,A}$: punctured GW-inv for type t, class $A \in NE(Y)$. $(g=0)$

k_t: a lattice index

Canonical wall structure for (Y, D): (picture due to Tim Gräfnitz)



VI. Step 4: X in codim ≤ 1

[$\mathbb{C}[[P]] = \varprojlim S_I$]

Work in finite approximation: $S_I = \mathbb{C}[\underset{NE(Y)}{\underset{\shortparallel}{P}}]/I$ Artinian

To construct X_I away from codim 2, glue copies of

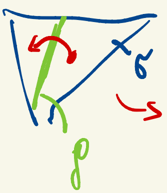
(I) $\operatorname{Spec} S_I[\sigma_{\mathbb{Z}}^{gp}] \simeq \mathbb{G}_m^n \times_{\mathbb{C}} \operatorname{Spec} S_I$, $\dim \sigma = n$

(II) $\operatorname{Spec} S_I[\varrho_{\mathbb{Z}}^{gp}][uv]/(uv - f_\rho \cdot t^{[Y_\varrho]})$, $\dim \varrho = n-1$

i.e. for walls in codim$=1$ cones of $\Sigma(Y)$ $\longrightarrow$ $\rho \subseteq \varrho$

<u>Gluing maps:</u>

- (I) $\hookrightarrow$ (II) open embedding if $\varrho \subseteq \sigma$ face
  [localize at u or v] $\leftarrow$ uses affine str. in codim$=1$!



- $\Theta_\rho : S_I[\sigma_{\mathbb{Z}}^{gp}] \circlearrowleft$, $\Theta_\rho(z^m) = \begin{cases} z^m, & m \in \rho_{\mathbb{Z}}^{gp} \\ f_\rho^{\pm 1} \cdot z^{\xi}, & \sigma_{\mathbb{Z}}^{gp}/\rho_{\mathbb{Z}}^{gp} = \mathbb{Z} \cdot \xi \end{cases}$

<u>Result:</u>

$$\begin{array}{c} \mathcal{U}_I \\ \downarrow \text{ flat} \\ \operatorname{Spec} S_I \end{array}$$

Later: $X_I \setminus \mathcal{U}_I \subseteq X_I$ has codim 2

V. Step 5: Extend over codim = 2

Abstractly: $\boxed{R_I := \Gamma(\mathcal{O}_{\mathfrak{U}_I})}$ (affine X) $\quad\left[QH^0(Y,D) = \varprojlim_{I} R_I \right]$

This works ($\mathfrak{U}_I \to \mathrm{Proj}\, R_I$ open embedding) because the ϑ_p on X_0 ($I = (P\setminus\{0\})$) extend to $\mathfrak{U}_I$ $\forall I$:

In a chart $\mathrm{Spec}\, S_I[\sigma_Z^{gp}]$ at $x \in \Sigma(Y) \setminus \bigcup$ walls:

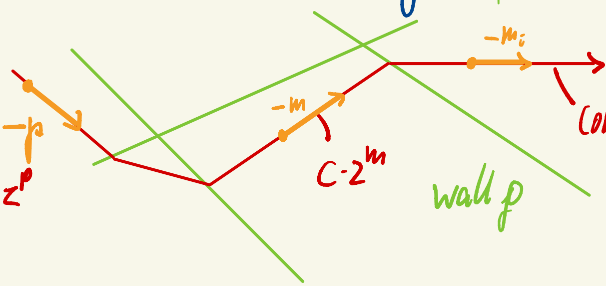
$$\boxed{\vartheta_p = \sum_{\tau} k_\tau N_{\tau,A} \cdot z^{-u_\tau} \cdot t^A \in S_I[\sigma_Z^{gp}]}$$

The sum runs over types τ of punctured tropical maps with two punctures with contact orders p and u_τ.

$N_{\tau,A}$: the corresponding $g=0$ punctured GW-inv., class $A \in NE(Y)$.

VI. Explicit computation via broken lines

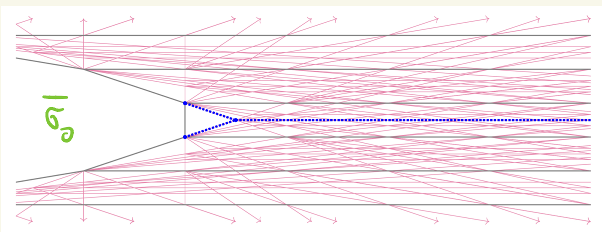
The <u>spine</u> of a punctured tropical map of type τ is a "broken line" – a piecewise linear path in $g^{-1}(1) \subseteq \Sigma(Y)$ carrying a monomial and bending at most when crossing walls:



continue with some $a_i z^{m_i}$

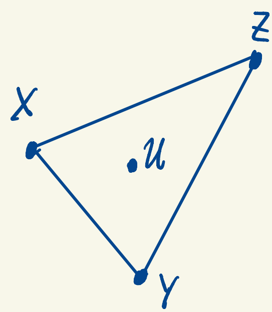
$$\vartheta_{\mathfrak{p}}(c \cdot z^m) = \sum_i a_i z^{m_i}$$

<u>Here:</u> No walls in bounded cell $\bar{\sigma}_0$:



- only toric relations among ϑ_p for $p \in \sigma_0(\mathbb{Z})$:

$$XYZ = u^3$$



- broken lines starting with $-e_4$ and ending in σ_0 can never bend:

$$\Rightarrow \quad W = \vartheta_{e_4} = -t \cdot \left(\frac{X}{u} + \frac{Y}{u} + \frac{Z}{u} \right)$$

VI. Outlook: Enumerative mirror symmetry

CY of dim $=3$, $P=N$.

Key: Have $\mathcal{X}^{\mathbb{R}} \subseteq \mathcal{X}$, often connected cmp. $\mathcal{X}^{>0} \subseteq \mathcal{X}^{\mathbb{R}}$

[$\forall t$: $\mathcal{X}^{>0} \approx B$, i.e. morally a section of the SYZ-fibration]

$$\int_{\mathcal{X}_t^{>0}} \Omega_t = \underset{\in \mathbb{Q}}{a} \cdot \log^3 t + \underset{\in \mathbb{Q}}{b} \cdot \log^2 t + h(t) \cdot \log t + g(t)$$

Both $h(t)$ and $g(t)$ contain the GW-potential.

[no mirror map correction since our families are already canonically parametrized]

Strategy: Compute via wall-structure and prove correspondence to log GW-theory of Y_0 ($=$ GW theory of Y_t, $t \neq 0$) via tropical interpretation & gluing formula in log-GW.

w/ van Garrel / Ruddat: works for $K\mathbb{P}^2$! (details in progress)

VII. Outlook: Homological mirror symmetry

w/ Pertz, J. Gaiter: (relative case, $D = Y_0$) fixed point Floer homology

$$QH^0(Y, Y_0) \xrightarrow[\simeq]{\log PSS} S(Y_s, \Phi) := \bigoplus_{p \geq 0} HF^0(Y_s, \Phi^p)$$

$\Phi : Y_{s \neq 0} \circlearrowleft$ symplectic monodromy

symplectic monodromy ring of $\begin{matrix} Y \\ \downarrow \\ \mathbb{C} \end{matrix}$

has a pair of pants product

Once this is established, and assuming the mirror X is generically smooth, HMS for maximally degenerating CY's follows from the following symplectic statement:

There exists a Lagrangian $L \subseteq Y_s$ that is a homology-sphere and s.th. the closed-open map [$L \Leftrightarrow$ section of SYZ-fibration]

$$\mathcal{CO} : S(Y_s, \Phi) \longrightarrow \bigoplus_{p \geq 0} HF^0(L, \Phi^p(L)) \quad \text{is an iso}$$

[e.g. a positive real locus $Y_s^{>0} \subseteq Y_s$ – exists if Y is itself an intrinsic mirror]

Then use Polishchuk's uniqueness theorem of A_∞-structures on $\bigoplus_{p,q \in \mathbb{Z}} H^q(X, \mathcal{O}_X(p))$ + automatic generation/core mirror symmetry (Abouzaid, Ganatra / Pertz/Sheridan)