UNIVERSITAT DE BARCELONA

Projective Absoluteness from Strong Cardinals

Cesare Straffelini

Universitat de Barcelona

Pisa, 26th October 2023

Generic Absoluteness

Generic Absoluteness is a property of models of Set Theory. The ultimate form of Generic Absoluteness would be something like:

for every forcing notion $\mathbb{P}$, V is elementarily equivalent to $V^{\mathbb{P}}$.

Generic Absoluteness is a property of models of Set Theory. The ultimate form of Generic Absoluteness would be something like:

for every forcing notion $\mathbb{P}$, V is elementarily equivalent to $V^{\mathbb{P}}$.

This is clearly impossible, since for example we can force both CH and $\neg$CH. We have three ways to weaken the statement.

1. Considering some model M, such as H_κ or $L(\mathbb{R})$, instead of V, and the elementary equivalence of M and $M^{\mathbb{P}}$.

1. Considering some model M, such as H_κ or $L(\mathbb{R})$, instead of V, and the elementary equivalence of M and $M^{\mathbb{P}}$.
2. Restricting the forcing notions to belong to some class Γ, for example the class of ccc, proper, stationary-preserving...

Generic Absoluteness

1. Considering some model M, such as H_κ or $L(\mathbb{R})$, instead of V, and the elementary equivalence of M and $M^{\mathbb{P}}$.
2. Restricting the forcing notions to belong to some class Γ, for example the class of ccc, proper, stationary-preserving...
3. Lowering the complexity of the elementary equivalence: instead of the whole equivalence, only for some formulæ.

Generic Absoluteness

1. Considering some model M, such as H_κ or $L(\mathbb{R})$, instead of V, and the elementary equivalence of M and $M^{\mathbb{P}}$.
2. Restricting the forcing notions to belong to some class Γ, for example the class of ccc, proper, stationary-preserving...
3. Lowering the complexity of the elementary equivalence: instead of the whole equivalence, only for some formulæ.

Obviously we can combine these weakenings.

Today, we are interested in generic absoluteness for the model V and for all forcing notions, but with **projective sentences**.

Generic Absoluteness

Today, we are interested in generic absoluteness for the model V and for all forcing notions, but with **projective sentences**.

A formula is Σ^1_0 if it is a first-order formula in the language of Peano Arithmetic. A formula is Π^1_n if its negation is Σ^1_n. A formula is Σ^1_{n+1} if it is $(\exists x \in \mathbb{R})\ \psi$, with ψ a Π^1_n-formula.

Generic Absoluteness

Today, we are interested in generic absoluteness for the model V and for all forcing notions, but with **projective sentences**.

A formula is Σ^1_0 if it is a first-order formula in the language of Peano Arithmetic. A formula is Π^1_n if its negation is Σ^1_n. A formula is Σ^1_{n+1} if it is $(\exists x \in \mathbb{R})\ \psi$, with ψ a Π^1_n-formula.

Σ^1_n-sentences (lightface) have no real parameters.
A sentence is $\boldsymbol{\Sigma}^1_n$ (boldface) if it is Σ^1_n with real parameters.

We will denote by $\mathrm{A}(\Phi)$ the statement:

for every forcing notion $\mathbb{P}$, V is Φ-elementarily equivalent to $V^{\mathbb{P}}$.

Generic Absoluteness

We will denote by $\mathrm{A}(\Phi)$ the statement:

for every forcing notion $\mathbb{P}$, V is Φ-elementarily equivalent to $V^{\mathbb{P}}$.

Shoenfield's Absoluteness theorem tells us that $\mathrm{A}(\boldsymbol{\Sigma}^1_2)$ holds.

We will denote by $A(\Phi)$ the statement:

for every forcing notion $\mathbb{P}$, V is Φ-elementarily equivalent to $V^{\mathbb{P}}$.

Shoenfield's Absoluteness theorem tells us that $A(\boldsymbol{\Sigma}^1_2)$ holds.

$A(\Sigma^1_3)$ is not provable in ZFC: the sentence «there exists a non-constructible real» is Σ^1_3, fails in L, and holds in $L[c]$ (Cohen).

We are interested in the **consistency strength** of

$$\mathrm{PA} :\equiv \textit{for all } n < \omega,\ \mathrm{A}(\boldsymbol{\Sigma}^1_n) \textit{ holds.}$$

We are interested in the **consistency strength** of

$$\mathsf{PA} :\equiv \textit{for all } n < \omega,\ \mathsf{A}(\boldsymbol{\Sigma}^1_n) \textit{ holds.}$$

Kai Hauser, *The Consistency Strength of Projective Absoluteness*: if ZFC + PA is consistent, then also ZFC + «there are ω-many strong cardinals» is consistent. And the other direction?

Strong Cardinals

A cardinal κ is λ-**strong** for some ordinal $\lambda \geqslant \kappa$ if there is $j : V \to M$ elementary embedding such that M is transitive, $V_\lambda \subseteq M$, the critical point of j is κ, and $j(\kappa) > \lambda$.

Strong Cardinals

A cardinal κ is λ-**strong** for some ordinal $\lambda \geqslant \kappa$ if there is $j : V \to M$ elementary embedding such that M is transitive, $V_\lambda \subseteq M$, the critical point of j is κ, and $j(\kappa) > \lambda$.

A cardinal κ is **strong** if it is λ-strong for all $\lambda \geqslant \kappa$.

Strong Cardinals

A cardinal κ is λ-**strong** for some ordinal $\lambda \geqslant \kappa$ if there is $j : V \to M$ elementary embedding such that M is transitive, $V_\lambda \subseteq M$, the critical point of j is κ, and $j(\kappa) > \lambda$.

A cardinal κ is **strong** if it is λ-strong for all $\lambda \geqslant \kappa$.

Any (κ-)strong cardinal κ is **measurable**: a characterization of measurable cardinals is «there is $j : V \to M$ with $\mathrm{crit}(j) = \kappa$».

Strong Cardinals

A famous theorem by Kunen tells that $0^\sharp$ exists if and only if there is a nontrivial elementary embedding $j : L \to L$.

A famous theorem by Kunen tells that $0^\sharp$ exists if and only if there is a nontrivial elementary embedding $j : L \to L$.

We can generalize Kunen's theorem and find that, for any set x, $x^\sharp$ exists if and only if there is a nontrivial elementary embedding $j : L(x) \to L(x)$. This could be a definition for «$x^\sharp$ exists».

A famous theorem by Kunen tells that $0^\sharp$ exists if and only if there is a nontrivial elementary embedding $j : L \to L$.

We can generalize Kunen's theorem and find that, for any set x, $x^\sharp$ exists if and only if there is a nontrivial elementary embedding $j : L(x) \to L(x)$. This could be a definition for «$x^\sharp$ exists».

Lemma

If there is a strong cardinal, then, for every set x, $x^\sharp$ exists.

Proof.

Let κ be strong, λ with $x \in V_\lambda$, and $j_1 : V \to M_1$ an elementary embedding with M_1 transitive, $V_\lambda \subseteq M_1$, $\mathrm{crit}(j_1) = \kappa$, $j_1(\kappa) > \lambda$.

Proof.

Let κ be strong, λ with $x \in V_\lambda$, and $j_1 : V \to M_1$ an elementary embedding with M_1 transitive, $V_\lambda \subseteq M_1$, $\mathrm{crit}(j_1) = \kappa$, $j_1(\kappa) > \lambda$. Since κ is measurable in V, and j_1 is elementary, $j_1(\kappa)$ is measurable in M_1. Let $j_2 : M_1 \to M_2$ be with $\mathrm{crit}(j_2) = j_1(\kappa)$.

Proof.

Let κ be strong, λ with $x \in V_\lambda$, and $j_1 : V \to M_1$ an elementary embedding with M_1 transitive, $V_\lambda \subseteq M_1$, $\text{crit}(j_1) = \kappa$, $j_1(\kappa) > \lambda$. Since κ is measurable in V, and j_1 is elementary, $j_1(\kappa)$ is measurable in M_1. Let $j_2 : M_1 \to M_2$ be with $\text{crit}(j_2) = j_1(\kappa)$. Since $x \in V_\lambda$, it holds $j_2(x) = x$. Then j_2 restricted to $L(x)$ is a nontrivial elementary embedding from $L(x)$ to $L(j_2(x)) = L(x)$. $\square$

Trees and Projections

A **tree** on $X^{<\omega} \times Y^{<\omega}$ is a set of pairs $\langle s, t \rangle$ with $s \in X^{<\omega}$, $t \in Y^{<\omega}$ and $\ell(s) = \ell(t)$, closed under initial segments.

A **tree** on $X^{<\omega} \times Y^{<\omega}$ is a set of pairs $\langle s, t\rangle$ with $s \in X^{<\omega}$, $t \in Y^{<\omega}$ and $\ell(s) = \ell(t)$, closed under initial segments.

The set of **infinite branches** $[T]$ of T is defined as

$$[T] := \{\langle x, y\rangle \in X^{\omega} \times Y^{\omega} : \forall n < \omega,\ \langle x \upharpoonright n, y \upharpoonright n\rangle \in T\}.$$

Trees and Projections

A **tree** on $X^{<\omega} \times Y^{<\omega}$ is a set of pairs $\langle s, t\rangle$ with $s \in X^{<\omega}$, $t \in Y^{<\omega}$ and $\ell(s) = \ell(t)$, closed under initial segments.

The set of **infinite branches** $[T]$ of T is defined as

$$[T] := \{\langle x, y\rangle \in X^\omega \times Y^\omega : \forall n < \omega,\ \langle x \restriction n, y \restriction n\rangle \in T\}.$$

The **projection** $p[T]$ of T is instead defined as

$$p[T] := \{x \in X^\omega :\ \exists y \in Y^\omega :\ \langle x, y\rangle \in [T]\}.$$

Notice that $p[T] = \varnothing$ if and only if $[T] = \varnothing$. In particular, $p[T] = \varnothing$ is absolute under set forcing: $p[T] = \varnothing$ in the ground model if and only if $p[T]^{V[G]} = \varnothing$ for any (or for all) generic G. Indeed, $[T] = \varnothing$ is equivalent to: $\langle T, \supseteq \rangle$ is well-founded.

Lemma

If S and T are trees on $X^{<\omega} \times Y^{<\omega}$ with $p[S] \cap p[T] = \varnothing$, then in any forcing extension $V[G]$ it holds $p[S]^{V[G]} \cap p[T]^{V[G]} = \varnothing$.

Trees and Projections

Let T be a tree on $\omega^{<\omega} \times \lambda^{<\omega}$. We define $T|_\delta$ as follows:

$$T|_\delta := \{\langle s, u \rangle \in T : u \in \delta^{<\omega}\}$$

for all $\omega < \delta < \lambda$.

Let T be a tree on $\omega^{<\omega} \times \lambda^{<\omega}$. We define $T|_\delta$ as follows:

$$T|_\delta := \{\langle s, u \rangle \in T : u \in \delta^{<\omega}\}$$

for all $\omega < \delta < \lambda$. If κ is strongly inaccessible, T is said to be κ-**good** if for all $\omega < \delta < \kappa$ with $\delta = \beth_\delta$ and every $\mathbb{P} \in V_\delta$,

$$\mathbb{P} \Vdash p[T] = p[T|_\delta].$$

Notice that the set of $\delta < \kappa$ with $\delta = \beth_\delta$ is a club in κ (since $\beth_\delta = |V_\delta|$). Every such δ is a strong limit: if $\mathbb{P} \in V_\delta$ then $|\mathbb{P}| < \delta$.

Lemma

If κ is strongly inaccessible, for all trees T on $\omega^{<\omega} \times \lambda^{<\omega}$ there is a κ-good tree T^* on $\omega^{<\omega} \times \kappa^{<\omega}$ such that $p[T] = p[T^*]$.

Lemma

If κ is strongly inaccessible, for all trees T on $\omega^{<\omega} \times \lambda^{<\omega}$ there is a κ-good tree T^* on $\omega^{<\omega} \times \kappa^{<\omega}$ such that $p[T] = p[T^*]$.

$\mathrm{Coll}(\omega, \delta)$, with δ a cardinal, is the partial order consisting of all finite sequences of ordinals less than δ, ordered by extension. This is the **Lévy collapse**, that forces δ to be countable in the generic extension. The cardinality of the forcing notion $\mathrm{Coll}(\omega, \delta)$ is δ.

Theorem (Woodin)

Let $\kappa < \lambda$ be cardinals such that κ is λ-strong and $\lambda = \beth_\lambda$.

Theorem (Woodin)

Let $\kappa < \lambda$ be cardinals such that κ is λ-strong and $\lambda = \beth_\lambda$. Let $j : V \to M$ be an elementary embedding with M transitive, $V_\lambda \subseteq M$, $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$. Let $T \subseteq \omega^{<\omega} \times \kappa^{<\omega}$ be κ-good.

Trees and Projections

Theorem (Woodin)

Let $\kappa < \lambda$ be cardinals such that κ is λ-strong and $\lambda = \beth_\lambda$. Let $j : V \to M$ be an elementary embedding with M transitive, $V_\lambda \subseteq M$, $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$. Let $T \subseteq \omega^{<\omega} \times \kappa^{<\omega}$ be κ-good.

If $G \subseteq \mathrm{Coll}(\omega, 2^{2^\kappa})$ is V-generic, then in $V[G]$ there is a tree S such that for every $\mathbb{P} \in V_\lambda[G]$ and every $V[G]$-generic filter $H \subseteq \mathbb{P}$,

$$V[G][H] \models p[j(T)] \sqcup p[S] = \mathbb{R}.$$

Universally Baire Sets

Universally Baire Sets

Universally Baire sets are a generalization of sets with the Baire property. They are very well-behaved: they are measurable, have the Baire property and (if $0^\sharp$ exists) the perfect set property.

Universally Baire sets are a generalization of sets with the Baire property. They are very well-behaved: they are measurable, have the Baire property and (if $0^\sharp$ exists) the perfect set property.

A set $A \subseteq \mathbb{R}$ is λ-**universally Baire** if for all compact Hausdorff spaces X with $|X| \leqslant \lambda$ and every continuous map $f : X \to \mathbb{R}$, the counterimage $f_{-1}A$ has the Baire property in X.

Universally Baire Sets

Universally Baire sets are a generalization of sets with the Baire property. They are very well-behaved: they are measurable, have the Baire property and (if $0^\sharp$ exists) the perfect set property.

A set $A \subseteq \mathbb{R}$ is λ-**universally Baire** if for all compact Hausdorff spaces X with $|X| \leqslant \lambda$ and every continuous map $f : X \to \mathbb{R}$, the counterimage $f_{-1}A$ has the Baire property in X.

A set is **universally Baire** if it is λ-uB for all cardinals λ.

Theorem (Feng-Magidor-Woodin)

A set $A \subseteq \mathbb{R}$ is λ-universally Baire if and only if there are trees S and T on $\omega^{<\omega} \times \omega^{<\omega}$ such that $A = p[S]$ and

$$\mathbb{P} \Vdash p[S] \sqcup p[T] = \mathbb{R}$$

for every forcing notion $\mathbb{P}$ with $|\mathbb{P}| \leqslant \lambda$.

Recall that a set $A \subseteq \mathbb{R}$ is said to be $\mathbf{\Sigma}^1_2$ if it's defined by a $\mathbf{\Sigma}^1_2$-formula, or equivalently if it's the continuous image of the complementary of an analytic subset of $\mathbb{R}$ (projective hierarchy).

Recall that a set $A \subseteq \mathbb{R}$ is said to be $\mathbf{\Sigma}^1_2$ if it's defined by a $\mathbf{\Sigma}^1_2$-formula, or equivalently if it's the continuous image of the complementary of an analytic subset of $\mathbb{R}$ (projective hierarchy).

Theorem (Feng-Magidor-Woodin)

The following are equivalent:

1. *for every set x, $x^\sharp$ exists;*
2. *every $\mathbf{\Sigma}^1_2$-subset of $\mathbb{R}$ is universally Baire.*

UB Sets and Absoluteness

UB Sets and Absoluteness

Every analytic ($\mathbf{\Sigma}^1_1$) subset of $\mathbb{R}$ is universally Baire (provably, in ZFC). The absoluteness $A(\mathbf{\Sigma}^1_2)$ is also provable in ZFC.

Every analytic ($\mathbf{\Sigma}^1_1$) subset of $\mathbb{R}$ is universally Baire (provably, in ZFC). The absoluteness $\mathrm{A}(\mathbf{\Sigma}^1_2)$ is also provable in ZFC.

Is there some link between these two notions?

UB Sets and Absoluteness

Every analytic ($\mathbf{\Sigma}^1_1$) subset of $\mathbb{R}$ is universally Baire (provably, in ZFC). The absoluteness $\mathrm{A}(\mathbf{\Sigma}^1_2)$ is also provable in ZFC.

Is there some link between these two notions?

Theorem

If every $\mathbf{\Sigma}^1_{n+1}$ subset of $\mathbb{R}$ is universally Baire, $\mathrm{A}(\mathbf{\Sigma}^1_{n+2})$ holds.

Proof.

Let φ be a $\mathbf{\Sigma}^1_{n+2}$-sentence. This means that

$$\varphi \equiv \exists x \in \mathbb{R}\ \forall y \in \mathbb{R}\ \psi(x, y, a)$$

where $\psi(x, y, z)$ is a Σ^1_n-formula and $a \in \mathbb{R}$ is a parameter.

Proof.

Let φ be a $\boldsymbol{\Sigma}^1_{n+2}$-sentence. This means that

$$\varphi \equiv \exists x \in \mathbb{R}\ \forall y \in \mathbb{R}\ \psi(x, y, a)$$

where $\psi(x, y, z)$ is a Σ^1_n-formula and $a \in \mathbb{R}$ is a parameter. Now, the set $A := \{x \in \mathbb{R} : \ \forall y \in \mathbb{R}\ \psi(x, y, a)\}$ is $\boldsymbol{\Pi}^1_{n+1}$.

Proof.

Let φ be a $\boldsymbol{\Sigma}^1_{n+2}$-sentence. This means that

$$\varphi \equiv \exists x \in \mathbb{R}\ \forall y \in \mathbb{R}\ \psi(x, y, a)$$

where $\psi(x, y, z)$ is a Σ^1_n-formula and $a \in \mathbb{R}$ is a parameter.
Now, the set $A := \{x \in \mathbb{R} : \ \forall y \in \mathbb{R}\ \psi(x, y, a)\}$ is $\boldsymbol{\Pi}^1_{n+1}$.
Fix a partial order $\mathbb{P}$. By hypothesis, there are trees S and T on $\omega^{<\omega} \times \omega^{<\omega}$ such that $A = p[S]$ and $\mathbb{P} \Vdash p[S] \sqcup p[T] = \mathbb{R}$.

Proof.

Let φ be a $\mathbf{\Sigma}^1_{n+2}$-sentence. This means that

$$\varphi \equiv \exists x \in \mathbb{R}\ \forall y \in \mathbb{R}\ \psi(x, y, a)$$

where $\psi(x, y, z)$ is a Σ^1_n-formula and $a \in \mathbb{R}$ is a parameter.
Now, the set $A := \{x \in \mathbb{R} : \ \forall y \in \mathbb{R}\ \psi(x, y, a)\}$ is $\mathbf{\Pi}^1_{n+1}$.
Fix a partial order $\mathbb{P}$. By hypothesis, there are trees S and T on $\omega^{<\omega} \times \omega^{<\omega}$ such that $A = p[S]$ and $\mathbb{P} \Vdash p[S] \sqcup p[T] = \mathbb{R}$.
Hence, φ is equivalent to $p[S] \neq \varnothing$, that is absolute. □

Lemma

Let M be a model of a fragment of ZFC, *let $\mathbb{P} \in M$, $H \subseteq \mathbb{P}$ be M-generic. Assume that every $\boldsymbol{\Sigma}^1_n$-set is uB in M and let $b \in \mathbb{R}^{M[H]}$ be a real. Then every $\boldsymbol{\Sigma}^1_n$-set is uB in $M[b]$.*

Lemma

Let M be a model of a fragment of ZFC, let $\mathbb{P} \in M$, $H \subseteq \mathbb{P}$ be M-generic. Assume that every $\boldsymbol{\Sigma}^1_n$-set is uB in M and let $b \in \mathbb{R}^{M[H]}$ be a real. Then every $\boldsymbol{\Sigma}^1_n$-set is uB in $M[b]$.

Theorem (Woodin)

Let $2 \leqslant n < \omega$ and let κ be a strong cardinal. Assume every $\boldsymbol{\Sigma}^1_n$-set is universally Baire. Then, after forcing with $\mathrm{Coll}(\omega, 2^{2^\kappa})$, in the forcing extension every $\boldsymbol{\Sigma}^1_{n+1}$-set is universally Baire.

Lemma

If κ is a strong cardinal, then, in $V^{\mathrm{Coll}(\omega,2^{2^\kappa})}$, $\mathrm{A}(\mathbf{\Sigma}^1_4)$ holds.

Lemma

If κ is a strong cardinal, then, in $V^{\mathrm{Coll}(\omega,2^{2^\kappa})}$, $\mathrm{A}(\boldsymbol{\Sigma}^1_4)$ holds.

Proof.

Since κ is strong, $x^\sharp$ exists for all sets x, hence every $\boldsymbol{\Sigma}^1_2$-set in V is universally Baire.

Lemma

If κ is a strong cardinal, then, in $V^{\mathrm{Coll}(\omega,2^{2^\kappa})}$, $\mathrm{A}(\boldsymbol{\Sigma}^1_4)$ holds.

Proof.

Since κ is strong, $x^\sharp$ exists for all sets x, hence every $\boldsymbol{\Sigma}^1_2$-set in V is universally Baire. Moreover, if $G \subseteq \mathrm{Coll}(\omega, 2^{2^\kappa})$ is V-generic, every $\boldsymbol{\Sigma}^1_3$-set in $V[G]$ is universally Baire.

Lemma

If κ is a strong cardinal, then, in $V^{\mathrm{Coll}(\omega,2^{2^\kappa})}$, $\mathrm{A}(\mathbf{\Sigma}^1_4)$ holds.

Proof.

Since κ is strong, $x^\sharp$ exists for all sets x, hence every $\mathbf{\Sigma}^1_2$-set in V is universally Baire. Moreover, if $G \subseteq \mathrm{Coll}(\omega, 2^{2^\kappa})$ is V-generic, every $\mathbf{\Sigma}^1_3$-set in $V[G]$ is universally Baire. Thus, $V[G] \models \mathrm{A}(\mathbf{\Sigma}^1_4)$. □

Clearly, we can replicate this reasoning and find

Lemma

If $\kappa_1 < \cdots < \kappa_n$ are strong cardinals, in $V^{\mathrm{Coll}(\omega, 2^{2^{\kappa_n}})}$ every $\boldsymbol{\Sigma}^1_{n+2}$-set is universally Baire and $\mathrm{A}(\boldsymbol{\Sigma}^1_{n+3})$ holds.

Clearly, we can replicate this reasoning and find

Lemma

If $\kappa_1 < \cdots < \kappa_n$ are strong cardinals, in $V^{\mathrm{Coll}(\omega, 2^{2^{\kappa_n}})}$ every $\mathbf{\Sigma}^1_{n+2}$-set is universally Baire and $\mathrm{A}(\mathbf{\Sigma}^1_{n+3})$ holds.

Theorem

If κ is the supremum of ω-many strong cardinals, then $V^{\mathrm{Coll}(\omega,\kappa)}$ is a model of Projective Absoluteness (PA). Hence,

$$\mathrm{Con}(\mathrm{ZFC} + \omega{-}\mathrm{SC}) \Rightarrow \mathrm{Con}(\mathrm{ZFC} + \mathrm{PA}).$$

Thanks for the attention!