Lie–Poisson discretization for incompressible magnetohydrodynamics on the sphere

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joint work with Klas Modin

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Exploiting Algebraic and Geometric Structure in Time-Integration Methods

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Incompressible MHD equations

$$\begin{cases} \dot{v} + \nabla_v v = -\nabla p + \mathrm{curl} B \times B, \\ \dot{B} = L_v B, \\ \mathrm{div} B = 0, \quad \mathrm{div} v = 0, \end{cases}$$

where $v(t, x)$ is the velocity, $B(t, x)$ is the magnetic field, $p(t, x)$ is the pressure.

- Energy (Hamiltonian):

$$H = \frac{1}{2} \int_{S^2} (|v|^2 + |B|^2) \mu$$

- Conserved quantities (Casimirs)
  - magnetic helicity: $C = \int_{S^2} \left(B, \mathrm{curl}^{-1}(B)\right) \mu,$.
  - cross-helicity: $J = \int_{S^2} (v, B) \mu$

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Vorticity formulation

One can introduce 4 scalar fields on S^2: Hamiltonians $\psi(t,x)$ and $\theta(t,x)$

$$B = X_\theta = \nabla^\perp \theta, \quad v = X_\psi = \nabla^\perp \psi,$$

and also vorticities $\beta(t,x)$ and $\omega(t,x)$

$$\omega = \mathrm{curl}(v) \cdot \mathbf{n}, \quad \beta = \mathrm{curl}(B) \cdot \mathbf{n}.$$

Proposition

Vorticity formulation for incompressible MHD equations is

$$\begin{cases} \dot\omega = \{\omega, \psi\} + \{\theta, \beta\}, & \omega = \Delta\psi, \\ \dot\theta = \{\theta, \psi\}, & \beta = \Delta\theta, \end{cases}$$

Continuous and quantized MHD

Quantization on the sphere: $(C_0^\infty, \{\cdot,\cdot\}) \approx (\mathfrak{su}(N), [\cdot,\cdot]_N)$

$$\omega, \theta \in C^\infty(S^2)$$

$$\begin{cases} \dot\omega = \{\omega, \Delta^{-1}\omega\} + \{\theta, \Delta\theta\}, \\ \dot\theta = \{\theta, \Delta^{-1}\omega\}, \end{cases}$$

- Magnetic helicity

$$C_f = \int_{S^2} f(\theta)\mu$$

- Cross-helicity

$$I_g = \int_{S^2} \omega g(\theta)\mu$$

$$W, \Theta \in \mathfrak{su}(N)$$

$$\begin{cases} \dot W = [W, \Delta_N^{-1} W] + [\Theta, \Delta_N \Theta], \\ \dot\Theta = [\Theta, \Delta_N^{-1} W], \end{cases}$$

- Magnetic helicity

$$C_f = \mathrm{tr}(f(\Theta))$$

- Cross-helicity

$$I_g = \mathrm{tr}(W g(\Theta))$$

Goal: time integrator that preserves Lie-Poisson geometry (in particular, Casimirs)

Geometric setting for quantized MHD

$$\begin{cases} \dot{W} = [W, M_1] + [\Theta, M_2], \\ \\ \dot{\Theta} = [\Theta, M_1], \end{cases} \tag{1}$$

where $M_1 = \Delta_N^{-1} W$, $M_2 = \Delta_N \Theta$.

Proposition

System (1) *is a Lie-Poisson flow on the dual* $\mathfrak{f}^*$ *of the Lie algebra* $\mathfrak{f} = \mathfrak{su}(N) \ltimes \mathfrak{su}(N)^*$*:*

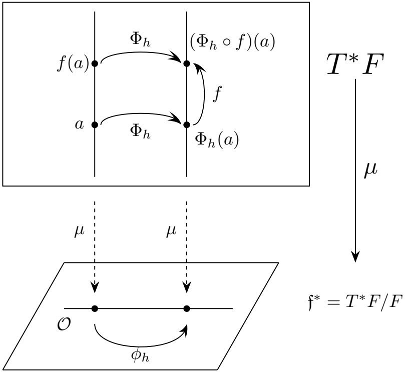
$$\dot{J} = \mathrm{ad}^*_M J,$$

where $J = (\Theta, W^\dagger) \in \mathfrak{f}^*$, $M = (M_1, M_2^\dagger) \in \mathfrak{f}$, *with the Hamiltonian*

$$H(W, \Theta) = \frac{1}{2} \left(\mathrm{tr}(W^\dagger M_1) + \mathrm{tr}(\Theta^\dagger M_2) \right). \tag{2}$$

Diagram

$$F = \mathrm{SU}(N) \ltimes \mathfrak{su}(N)^*, \quad \mathfrak{f} = \mathfrak{su}(N) \ltimes \mathfrak{su}(N)^*$$



Coadjoint orbit $\mathcal{O} = \{\text{level set of Casimirs}\}$

K. Modin, M. Viviani, Lie–Poisson Methods for Isospectral Flows. Found. Comput. Math. 20. 889-921 (2020).

Momentum map and other animals

$$F = \mathrm{SU(N)} \ltimes \mathfrak{su}(N)^*$$

- Cotangent bundle T^*F,

$$T^*F = \left\{(Q, m, P, \alpha) \mid Q \in \mathrm{SU}(N), P \in T^*_Q(\mathrm{SU}(N)), m \in \mathfrak{su}(N)^*, \alpha \in \mathfrak{su}(N)\right\}$$

- Left lifted action of $F \ni (G, u)$

$$(G, u) \cdot (Q, m, P, \alpha) = (GQ, \mathrm{Ad}^*_Q u + m, (G^{-1})^\dagger P, \alpha)$$

- Momentum map $\mu\colon T^*F \to \mathfrak{f}^*$

$$\mu(Q, m, P, \alpha) = \left(\frac{PQ^\dagger - QP^\dagger}{2}, Q\alpha Q^\dagger\right) = (W^\dagger, \Theta).$$

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Canonical equations on T^*F

Proposition

Canonical equations on T^*F

$$\begin{cases} \dot{Q} = -M_1 Q, \\ \dot{P} = M_1^\dagger P + 2M_2^\dagger Q\alpha^\dagger, \\ \dot{\alpha} = 0, \end{cases}$$

where $M_1 = \Delta_N^{-1} W, \quad M_2 = \Delta_N \Theta$, *with right-invariant Hamiltonian* $\tilde{H} = H \circ \mu$, *are reduced to the Lie-Poisson system on* $\mathfrak{f}^*$

$$\dot{W} = [W, M_1] + [\Theta, M_2], \quad \dot{\Theta} = [\Theta, M_1],$$

by means of the momentum map

$$\mu(Q, m, P, \alpha) = \left(\frac{PQ^\dagger - QP^\dagger}{2}, Q\alpha Q^\dagger \right) = (W^\dagger, \Theta).$$

Implicit midpoint method

Hamiltonian system on T^*F

$$\begin{cases} \dot{Q} = H_P(P,Q), \\ \dot{P} = -H_Q(P,Q). \end{cases}$$

Implicit midpoint method

$$\begin{cases} Q_n = \tilde{Q} - \dfrac{h}{2} H_P(\tilde{P},\tilde{Q}), \\ \\ P_n = \tilde{P} + \dfrac{h}{2} H_Q(\tilde{P},\tilde{Q}) \end{cases} \qquad \begin{cases} Q_{n+1} = \tilde{Q} + \dfrac{h}{2} H_P(\tilde{P},\tilde{Q}), \\ \\ P_{n+1} = \tilde{P} - \dfrac{h}{2} H_Q(\tilde{P},\tilde{Q}) \end{cases}$$

- symplectic
- equivariant with respect to the right lifted action of $F = \mathrm{SU}(N) \ltimes \mathfrak{su}(N)^*$ on T^*F.

Structure preserving scheme for MHD

Implicit midpoint scheme descends to an integrator on $\mathfrak{f}^*$

$$\begin{cases} W_n = \tilde{W} - \frac{h}{2}[\tilde{W}, \tilde{M_1}] - \frac{h}{2}[\tilde{\Theta}, \tilde{M_2}] - \frac{h^2}{4}\left(\tilde{M_1}\tilde{W}\tilde{M_1} + \tilde{M_2}\tilde{\Theta}\tilde{M_1} + \tilde{M_1}\tilde{\Theta}\tilde{M_2}\right), \\ \Theta_n = \tilde{\Theta} - \frac{h}{2}[\tilde{\Theta}, \tilde{M_1}] - \frac{h^2}{4}\tilde{M_1}\tilde{\Theta}\tilde{M_1} \end{cases}$$

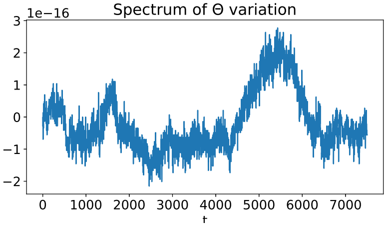
$$\begin{cases} W_{n+1} = W_n + h[\tilde{W}, \tilde{M_1}] + h[\tilde{\Theta}, \tilde{M_2}], \\ \Theta_{n+1} = \Theta_n + h[\tilde{\Theta}, \tilde{M_1}]. \end{cases} \tag{3}$$

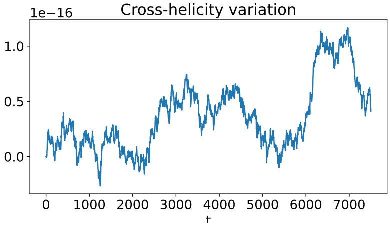
Theorem (K. Modin, MR)

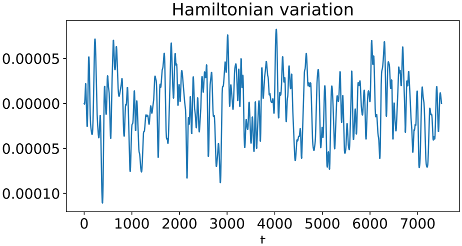
The numerical scheme (3) *is a Lie-Poisson integrator* $\phi_h : \mathfrak{f}^* \to \mathfrak{f}^*$, $\phi_h : (W_n, \Theta_n) \mapsto (W_{n+1}, \Theta_{n+1})$ *for ideal MHD equations on the sphere. The integrator* (3) *preserves Casimirs:*

$$\mathrm{tr}(f(\Theta_n)) = \mathrm{tr}(f(\Theta_{n+1})), \quad \mathrm{tr}(W_n g(\Theta_n)) = \mathrm{tr}(W_{n+1} g(\Theta_{n+1})).$$

Simulation


Spectrum of Θ variation
1e−16
3
2
1
0
−1
−2
0
1000
2000
3000
4000
5000
6000
7000
t


Cross-helicity variation
1e−16
1.0
0.5
0.0
0
1000
2000
3000
4000
5000
6000
7000
t


Hamiltonian variation
0.00005
0.00000
0.00005
0.00010
0
1000
2000
3000
4000
5000
6000
7000
t

Alfvén wave turbulence equations (Hazeltine's model)[1]

$$\begin{cases} \dot{\omega} = \{\omega, \Delta^{-1}\omega\} + \{\theta, \Delta\theta\}, \\ \dot{\theta} = \{\theta, \Delta^{-1}\omega\} - \alpha\{\theta, \chi\}, \\ \dot{\chi} = \{\chi, \Delta^{-1}\omega\} + \{\theta, \Delta\theta\}, \end{cases}$$

where ω and θ have the same meaning as before, χ is the normalized deviation of particle density from a constant equilibrium value, α is a constant parameter.
Hamiltonian is

$$H = \frac{1}{2} \int_{S^2} \left(\omega\Delta^{-1}\omega + \theta\Delta\theta - \alpha\chi^2\right) \mu,$$

and Casimirs are

$$\mathcal{C} = \int_{S^2} \left(f(\theta) + \chi g(\theta) + k(\omega - \chi)\right) \mu$$

for arbitrary smooth functions f, g, k.

[1]R.D. Hazeltine, D. Holm, P.J. Morrison. Electromagnetic solitary waves in magnetized plasmas. J. Plasma Phys. 34(1), 103-114 (1985).

Quantized Hazeltine's equations

Using the geometric quantization approach, we get a spatially discretized analogue of Hazeltine's equations in the form

$$\begin{cases} \dot{W} = [W, M_1] + [\Theta, M_2], \\ \dot{\Theta} = [\Theta, M_1] - \alpha[\Theta, \chi], \\ \dot{\chi} = [\chi, M_1] + [\Theta, M_2], \end{cases} \iff \begin{cases} \dot{\Psi} = [\Psi, M_1], \\ \dot{\Theta} = [\Theta, M_3], \\ \dot{\chi} = [\chi, M_3] + [\Theta, M_2], \end{cases} \tag{4}$$

where $W, \Theta, \chi \in \mathfrak{su}(N)$, $\Psi = W - \chi$, $M_1 = \Delta_N^{-1} W$, $M_2 = \Delta_N \Theta$, $M_3 = M_1 - \alpha\chi$.

Proposition

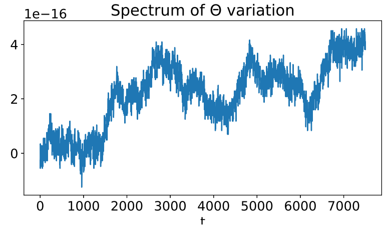
System (4) is a Lie-Poisson flow on the dual $\mathfrak{f}^*$ of the Lie algebra

$$\mathfrak{f} = \mathfrak{su}(N) \oplus (\mathfrak{su}(N) \ltimes \mathfrak{su}(N)^*) .$$

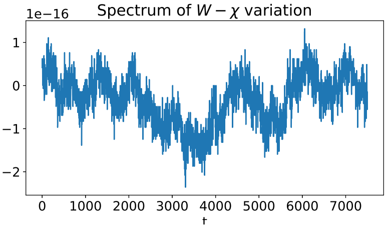
with Casimirs

$$\mathcal{E}_k = \mathrm{tr}(k(W - \chi)), \quad \mathcal{C}_f = \mathrm{tr}(f(\Theta)), \quad J_g = \mathrm{tr}(\chi g(\Theta)).$$

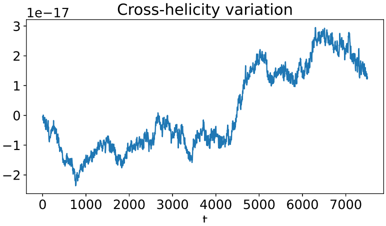
Simulation



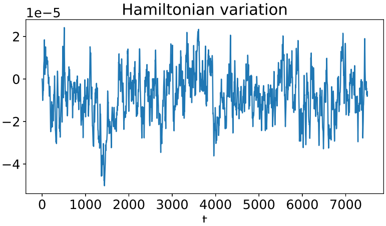
Spectrum of Θ variation
1e−16
4
2
0
0
1000
2000
3000
4000
5000
6000
7000
t



Spectrum of $W - \chi$ variation
1e−16
1
0
−1
−2
0
1000
2000
3000
4000
5000
6000
7000
t



Cross-helicity variation
1e−17
3
2
1
0
−1
−2
0
1000
2000
3000
4000
5000
6000
7000
t



Hamiltonian variation
1e−5
2
0
−2
−4
0
1000
2000
3000
4000
5000
6000
7000
t

MHD: different regimes

- Total energy

$$H = \frac{1}{2} \int_{S^2} \left(\omega \Delta^{-1} \omega + \theta \Delta \theta\right) \mu = H_\omega + H_\theta$$

- Weak magnetic field

$$H_\theta \ll H_\omega$$

- Generic magnetic field

$$H_\theta \approx H_\omega$$

MHD dynamics: weak magnetic field

Click!

W field

Click!

Θ field

MHD dynamics: generic magnetic field

Click!

W field

Click!

Θ field

Hazeltine's model: different regimes

- Total energy

$$H = \frac{1}{2}\int_{S^2} \left(\omega\Delta^{-1}\omega + \theta\Delta\theta - \alpha\chi^2\right)\mu = H_\omega + H_\theta + H_\chi$$

- Weak magnetic field

$$H_\theta \ll H_\omega \approx H_\chi$$

- Generic magnetic field

$$H_\theta \approx H_\omega \approx H_\chi$$

Hazeltine's model, weak magnetic field, zero W momentum

Click!

W field

Click!

Θ field

Click!

χ field

Hazeltine's model, generic magnetic field

Click!

W field

Click!

Θ field

Click!

χ field

Generic magnetic field, Θ dynamics, initial mixing

Click!

Magnetic field Θ dynamics

Generic magnetic field, Θ dynamics, dipole

Click!

Magnetic field Θ dynamics, dipole dynamics

Thank you for attention!

Quantization on the sphere: $(C^\infty, \{\cdot,\cdot\}) \approx (\mathfrak{u}(N), [\cdot,\cdot]_N)$

Laplace-Beltrami operator

$$\Delta(\cdot) = \sum_{k=1}^{3} \{x_k, \{x_k, \cdot\}\}$$

Y_{lm} are eigenvectors of Δ

$$\Delta Y_{lm} = -l(l+1) Y_{lm}$$

Hoppe-Yau Laplacian

$$\Delta_N(\cdot) = \sum_{k=1}^{3} [X_k, [X_k, \cdot]_N]_N$$

T^N_{lm} are eigenvectors of Δ_N

$$\Delta_N T^N_{lm} = -l(l+1) T^N_{lm},$$

Vorticity function

$$\omega = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \omega^{lm} Y_{lm}$$

$$\xrightarrow{p_N}$$

Vorticity matrix

$$W = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} \omega^{lm} T^N_{lm}$$

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Vorticity function

Vorticity matrix

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Vorticity matrix

$$W = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} \omega^{lm} T^N_{lm}$$

What are T^N_{lm}?

$$T^N_{lm} = \begin{pmatrix} 0 & \dots & 0 & \times & 0 & \dots & 0 \\ & \ddots & & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & & \ddots & \ddots & 0 \\ & & & \ddots & & \ddots & \times \\ & \Large{0} & & & \ddots & & 0 \\ & & & & & \ddots & \vdots \\ & & & & & & 0 \end{pmatrix}$$

$(\times, \dots, \times) = m -$ off diagonal

Convergence results

Theorem (Charles, Polterovich (2017))

For every $f, g \in C^\infty(\mathrm{S}^2)$ *there exist* $c_0, c_1 > 0$, *such that*

- $\|f\|_{L^\infty} - \dfrac{c_0\|f\|_{C^2}}{N} \leq \|p_N(f)\|_{L^\infty_N} \leq \|f\|_{L^\infty}$

- $\|[p_N(f), p_N(g)]_N - p_N(\{f, g\})\|_{L^\infty_N} \leq$
  $\dfrac{c_1}{N}\left(\|f\|_{C^1}\|g\|_{C^3} + \|f\|_{C^2}\|g\|_{C^2} + \|f\|_{C^3}\|g\|_{C^1}\right),$

where

$$\|f\|_{C^k} = \max_{i \leq k} \sup |\nabla^i f|, \quad \|A\|_{L^\infty_N} = \sup_{\|x\|=1} \|Ax\|, \quad A \in \mathfrak{u}(N)$$

.