Groups with cofinite Zariski topology and potential density

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Potentially dense sets of a group

Example (Markov, Dokl. AN SSSR 1944)

If $A = \{a_n : n \in \mathbb{N}\}$ is an infinite subset of $\mathbb{Z}$, then by Weyl's uniform equidistribution theorem there exists an irrational $\alpha \in [0, 1]$ such that the set $\{a_n\alpha : n \in \mathbb{N}\}$ is dense mod 1 in $\mathbb{R}$.
Then for the injective homomorphism $f : \mathbb{Z} \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$ defined by $f(n) = n\alpha$ the set $f(A)$ is dense in $\mathbb{T}$ (so in $f(\mathbb{Z})$ as well).
If τ_α is the topology induced on $\mathbb{Z}$ by this embedding, then A is dense in $(\mathbb{Z}, \tau_\alpha)$.

Definition (Markov Dokl. AN SSSR 1944)

A subset X of a group G is *potentially dense* if X is dense in *some* Hausdorff group topology on G.

Problem (Markov Dokl. AN SSSR 1944)

(a) Describe the potentially dense subsets of a group G.
(b) Describe the class $\mathcal{P}$ of groups G in which every infinite subset is potentially dense.

Unconditionally closed sets and algebraic sets of a group

Definition (Markov, Dokl. AN SSSR 1944)

A subset X of a group G is *unconditionally closed* if X is closed in *every* Hausdorff group topology on G.

The family of all unconditionally closed sets in a group G, is stable under taking finite unions and arbitrary intersections, so it is the family of all closed sets of a T_1 topology $\mathfrak{M}_G$ on G, named *Markov topology* of G. The potentially dense sets are $\mathfrak{M}_G$-dense.

Example

Let $w(x) = g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \cdots g_n x^{\varepsilon_n}$ be a word in $G * \langle x \rangle$, where x is a variable, $n \geq 0$, $g_1, \ldots, g_n \in G$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$. The set $E_w^G = \{g \in G : w(g) = e_G\}$ is called *elementary algebraic*.
The elementary algebraic sets and their finite unions are unconditionally closed. The same holds true for the intersections of finite unions of elementary algebraic (called *algebraic*) subsets of G.

The Zariski topology

The Zariski topology $\mathfrak{Z}_G$ of a group G has the family $\mathfrak{A}_G$ of algebraic sets of G as the family of $\mathfrak{Z}_G$-closed sets. It was introduced explictly by R. Bryant under the name verbal topology. As algebraic sets are unconditionally closed, one has $\mathfrak{Z}_G \leq \mathfrak{M}_G$. Markov proved that $\mathfrak{Z}_G = \mathfrak{M}_G$ for countable groups, Perel′man - for abelian groups, Shakhmatov–Yanez for free groups [2023] (but $\mathfrak{Z}_G \neq \mathfrak{M}_G$ in general – Hesse's PhD thesis [1978]).

Example

1. If G is abelian, an elementary algebraic set has the form $E_w = g + G[n]$, where $G[n] = \{g \in G : ng = 0\}$ and $\mathfrak{A}_G = \{\emptyset\} \cup \{\bigcup_{i=1}^m g_i + G[n_i] \mid m \in \mathbb{N}_+, g_i \in G, n_i \in \mathbb{N}\}$.
2. Centralizers are elementary algebraic sets, so they are Zariski-closed along with the center $Z(G)$.
3. Banakh, Guran and Protasov [2012] proved that if $G = Sym(X)$ is the symmetric group over any set X, then $\mathfrak{Z}_G = \mathfrak{M}_G$ and they coincide with the pointwise convergence topology of $Sym(X) \leq X^X$ (carrying the product topology).

When $\mathfrak{Z}_G$ is a group topology: $\mathfrak{Z}$-groups

Call a group G a $\mathfrak{Z}$-group if $\mathfrak{Z}_G$ is a group topology. Clearly, $\mathfrak{Z}_G$ (being T_1) is Hausdorff if G is a $\mathfrak{Z}$-group.
For a topological space X denote by $\mathcal{H}(X)$ the group of all homeomorphisms of X. So, for a discrete X, $\mathcal{H}(X) = Sym(X)$ is a $\mathfrak{Z}$-group, by (3). This idea inspires other examples:

Example (more $\mathfrak{Z}$-groups: Chang & Gartside, Merelishvili & Polev)

When $X = [0,1]$ or $X = S^1$ (the unit circle), then $\mathcal{H}(X)$ is a $\mathfrak{Z}$-group ($\mathfrak{Z}_{\mathcal{H}(X)}$ coincides with the compact-open topology).
This remains true if we replace $[0,1]$ or S^1 by any of the following:

- metric one-dimensional manifold (with or without boundary);
- a compact connected ordered space such that for every pair $a < b$ in X there exist $c, d \in X$ with $a \leq c < d \leq b$ and $[c,d]$ is separable:
- a compact metrizable space, such that the set of points $x \in X$ having a neighborhood homeomorphic to $(0,1)$ is dense in X.

Groups which satisfy min-closed

Bryant [J. Algebra 1977] studied the class $\mathcal{N}$ of groups with the minimal condition on Zariski closed sets (groups which satisfy *min*-closed, $\mathcal{N}$ stays for Noetherian) and provided these examples

Example ($\mathcal{N}$ contains the following classes of groups:)

(a) all linear groups over arbitrary fields.

(b) all finitely generated, abelian-by-nilpotent-by-finite group;

(c) all abelian-by-finite groups.

Bryant [J. Algebra 1977] proved that the class $\mathcal{N}$ is stable under taking subgroups, and under taking finite products.

Theorem (Toller, DD 2012)

(a) *A group $G \in \mathcal{N}$ iff every countable subgroup of G is in $\mathcal{N}$. Hence, $\mathcal{N}$ contains all free groups.*

(b) *If $G = \prod_{i \in I} G_i$, then $G \in \mathcal{N}$ iff every $G_i \in \mathcal{N}$ and all but finitely many of the groups G_i are abelian.*

The class $\mathcal{C}$ of groups G with cofinite $\mathfrak{Z}_G$

The first main topic of this talk is the class $\mathcal{C}$ of groups G with $\mathfrak{Z}_G = \text{cofin}_G$ – the cofinite topology of G. Similarly, let $\mathcal{M}$ be the class of groups G with $\mathfrak{M}_G = \text{cofin}_G$.
Obviously $\mathcal{M} \subseteq \mathcal{C}$ (as $\mathfrak{Z}_G \leq \mathfrak{M}_G$), while obviously $\mathcal{C} \subseteq \mathcal{N}$ (since cofin_G is Noetherian). Therefore, $\mathcal{M} \subseteq \mathcal{C} \subseteq \mathcal{N}$.
On the other hand, $G \in \mathcal{M}$ iff all proper unconditionally closed sets of G are finite, therefore one has $\mathcal{P} \subseteq \mathcal{M} \subseteq \mathcal{C} \subseteq \mathcal{N}$ (while $\mathcal{P}$ vacuously contains all finite groups).

All groups from Bryant's series of examples in $\mathcal{N}$ belong to $\mathcal{C}$ iff they are finite, therefore the inclusion $\mathcal{C} \subset \mathcal{N}$ is proper.

Finally, the abelian groups in $\mathcal{M}$ and $\mathcal{C}$ are the same, as $\mathfrak{Z}_G = \mathfrak{M}_G$ when G is abelian. So $\mathcal{M} = \mathcal{C}$, modulo the following conjecture

Main Conjecture

The groups in $\mathcal{C}$ are always either finite or abelian.

Complete description of $\mathcal{P}$ in the abelian case

Call a group G almost torsion-free if G has only finitely many elements of order p for every prime p.

Lemma (Tkachenko and Yaschenko, Topology Appl. 2002)

If G is an abelian group, then $G \in \mathcal{C}$ iff G is either of prime exponent or G is almost torsion-free.

If G is an almost torsion-free abelian group, then infinite subset of G is $\mathfrak{Z}$-dense, while every potentially dense set is also $\mathfrak{Z}$-dense. They proved a partial inverse of the latter implication:

Theorem (Tkachenko and Yaschenko, Topology Appl. 2002)

If G is an almost torsion-free abelian group with $|G| \leq \mathfrak{c}$, then $G \in \mathcal{P}$ (i.e., $\mathfrak{Z}$-dense $\to$ potentially dense).

They asked whether $|G| \leq \mathfrak{c}$ can be relaxed to $|G| \leq 2^{\mathfrak{c}}$. The necessity of the restraint $|G| \leq 2^{\mathfrak{c}}$ is a well known fact (a Hausdorff space with a dense countable subset has size $\leq 2^{\mathfrak{c}}$).

Simultaneous realization for the Zariski closure

Theorem (Shakhmatov, DD, Adv. Math. 2011)

Let G be an abelian group with $|G| \leq 2^{\mathfrak{c}}$, and let $\mathcal{X}$ be a countable family of subsets of G. Then there exists a Hausdorff group topology τ on G such that the τ-closure of each $X \in \mathcal{X}$ coincides with its Zariski closure.

In particular, if $\mathcal{X}$ is a countable family of $\mathfrak{Z}$-dense subsets of G, then their potential density is simultaneously witnessed by a single Hausdorff group topology on G. If G has a potentially dense countable set then $|G| \leq 2^{\mathfrak{c}}$, so this inequality is a natural restraint:

Corollary (Shakhmatov, DD, Adv. Math. 2011)

(a) *A countable subset S of an abelian group G is potentially dense if and only if S is $\mathfrak{Z}$-dense and $|G| \leq 2^{\mathfrak{c}}$.*

(b) *If $G \in \mathcal{C}$ is abelian, then $G \in \mathcal{P}$ if and only if $|G| \leq 2^{\mathfrak{c}}$.*

If our conjecture holds true, "abelian" can be omitted in item (b).

Potential density of uncountable subsets of abelian groups

In an abelian group G one has the following immediate necessary condition for potential density of a subset $S \subseteq G$:

$$|nG| \leq 2^{2^{|nS|}} \quad \text{for all } n \in \mathbb{N} \setminus \{0\}. \tag{1}$$

A slightly reinforced version of (1) is a sufficient condition for potential density of an uncountable subset S:

Theorem (Shakhmatov – DD, Proc. AMS 2010)

If S is an uncountable subset of an abelian group G such that $|G| \leq 2^{2^{|nS|}}$ for all $n \in \mathbb{N} \setminus \{0\}$, then S is potentially dense.

The gap between the above sufficient condition and the necessary condition (1) obviously disappears for groups satisfying $|nG| = |G|$ for every $n \in \mathbb{N}_+$. For such groups we get a necessary and sufficient condition for potential density of an uncountable subset S.

Among the groups satisfying $|nG| = |G|$ for every integer $n \geq 1$ are all divisible groups as well as the groups G such that $|t(G)| < |G|$ (in particular, all almost torsion-free groups).

Partial Zariski topologies: $\mathfrak{Z}_{mon}$ and the centralizer topology

For a set of words $\mathcal{W}$ we consider the topology $\mathfrak{T}_{\mathcal{W},G}$ on G having the family of elementary algebraic subsets $\{E_w \mid w \in \mathcal{W}\}$ as a subbase of the closed sets of $\mathfrak{T}_{\mathcal{W}}$. Such a topology is called a partial Zariski topology, as obviously $\mathfrak{T}_{\mathcal{W}} \leq \mathfrak{Z}_G$.

Example (partial Zariski topologies)

a) $\mathrm{cofin}_G = \mathfrak{T}_{\{gx \mid g \in G\}}$, as $E_{gx} = \{g^{-1}\}$ for every $g \in G$.

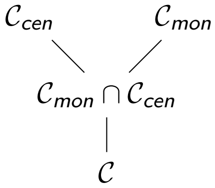
b) With $\mathcal{M} = \{gx^n \mid g \in G,\ n \in \mathbb{N}\}$, one gets the monomial topology $\mathfrak{Z}_{mon,G} := \mathfrak{T}_{\mathcal{M}}$. If G is abelian, then $\mathfrak{Z}_{mon,G} = \mathfrak{Z}_G$.

c) With $\mathcal{W} = \{hxgx^{-1} \mid g, h \in G\}$, $\mathfrak{C}_G = \mathfrak{T}_{\mathcal{W},G}$ is the centralizer topology of G (as $\{E_{hxgx^{-1}} :\mid g, h \in G\}$ is the family of all cosets of centralizers of elements of G). $\mathfrak{C}_G$ need not be T_1.

If G is a free non-abelian group or the Heisenberg group $\mathbb{H}_K$ of 3×3 unitriangular matrices over a field K with char $K = 0$, then $\mathrm{cofin}_G < \mathrm{cofin}_G \vee \mathfrak{C}_G = \mathfrak{Z}_G$, while $\mathfrak{C}_G \not\leq \mathrm{cofin}_F$ (as G has infinite centralizers), so $G \notin \mathcal{C}_{cen}$ – the class of groups with $\mathfrak{C}_G \leq \mathrm{cofin}_F$.

The classes $\mathcal{C}_{mon}$ and $\mathcal{C}_{cen}$

Let $\mathcal{C}_{mon}$ (resp., $\mathcal{C}_{cen}$) be the class of groups G with $\mathfrak{Z}_{mon,G} = \mathrm{cofin}_G$ (resp., $\mathfrak{C}_G \leq \mathrm{cofin}_F$). Then $\mathcal{C}_{mon} \cap \mathcal{C}_{cen} \supseteq \mathcal{C}$.



Theorem

- (a) *$\mathcal{C}$, $\mathcal{C}_{cen}$, and $\mathcal{C}_{mon}$ are stable with respect to taking subgroups and quotients with respect to finite normal subgroups.*
- (b) *If G is an infinite group then $G \in \mathcal{C}$ (resp., $G \in \mathcal{C}_{cen}$, $G \in \mathcal{C}_{mon}$) iff $H \in \mathcal{C}$ (resp., $H \in \mathcal{C}_{cen}$, $H \in \mathcal{C}_{mon}$) for every countable subgroup H of G.*

Description of $\mathcal{C}_{mon}$

Recall that a group G is said to satisfy the cancellation law if $x^n = y^n$ implies $x = y$, for every $n \in \mathbb{N}_+$ and $x, y \in G$.

Definition

A group G is said to satisfy the *Weak Cancellation Law* (shortly, *WCL*) if for every $n > 0$ the map $x \mapsto x^n$ in G is finite-to-one.

Clearly, WCL implies "almost torsion-free". This notion is quite useful for complete understanding of the equality $\mathfrak{Z}_{mon} = \mathfrak{Z}_G = \mathfrak{M}_G$ when G is abelian, (as WLC coincides with "almost torsion-free" for abelian groups):

Theorem

An infinite group G is $\mathfrak{Z}_{mon}$-cofinite if and only if either G has prime exponent, or G is WCL.

If G is a free non-abelian group or the Heisenberg group $\mathbb{H}_K$ over a field K with $\mathrm{char}K = 0$, then G satisfies WCL, thus $G \in \mathcal{C}_{mon} \setminus \mathcal{C}_{cen}$.

Properties of the classes $\mathcal{C}$, $\mathcal{C}_{mon}$ and $\mathcal{C}_{cen}$

Proposition

If $G \in \mathcal{C}_{cen}$ is infinite non-abelian, then
(a) $Z(G)$ is finite;
(b) $G^{(n)}$ is infinite for every n, so G has no infinite solvable subgroups and G is torsion.
(c) if H is a finite subgroup of G, then either $N_G(H)$ is finite, or $H \leq Z(G)$. In particular, H is normal if and only if it is central.
(d) the infinite subgroups of G are neither locally finite, nor locally solvable.

Theorem

If $G \in \mathcal{C}_{mon} \cap \mathcal{C}_{cen}$ is infinite non-abelian, then it has a prime exponent $p \geq 5$.

Corollary

If $G \in \mathcal{C}$ is infinite non-abelian, then G has a prime exponent $p \geq 5$. If $g \in G \setminus Z(G)$, then $N_G(\langle g \rangle) = C_G(g)$ is finite.

Maximal finite subgroups of $G \in \mathcal{C}_{cen}$ or $G \in \mathcal{C}$

Proposition

If $G \in \mathcal{C}_{cen}$ is infinite non-abelian, then every finite subgroup of G is contained in a maximal finite subgroup M satisfying

$$Z(G) \leq Z(M) = C_G(M) \leq N_G(M) = M.$$

Proposition

Let $G \in \mathcal{C}$ be an infinite group.

(a) *Then $M_1 \cap M_2 = Z(G)$ for every pair of maximal finite subgroups $M_1 \neq M_2$ of G.*

(b) *Every element $g \in G \setminus Z(G)$ in contained in a unique maximal finite subgroup M_g of G, and $N_G(\langle g \rangle) \leq M_g$.*

(c) *If $x, y \in G$ do not belong to the same maximal finite subgroup of G, then $H = \langle x, y \rangle$ is infinite and $Z(H) = Z(G) \cap H$. In particular, $|Z(H)| = 1$, provided $|Z(G)| = 1$.*

Summary

In the chase for infinite non-abelian groups in the class $\mathcal{C}$, one can assume several additional properties.

Theorem

If there exists an infinite non-abelian group $G \in \mathcal{C}$, then there exists an infinite group $K \in \mathcal{C}$ such that:

(a) *K is finitely generated and has prime exponent,*
(b) *K is perfect, center-free and indecomposable,*
(c) *K has no proper subgroups of finite index,*
(d) *K has no proper finite normal subgroups.*

Every infinite finitely generated simple group of prime exponent satisfies (a)–(d), in particular so does every Tarski Monster (seemingly, none of the Tarski Monsters built so far belongs to $\mathcal{C}$).

Open questions

We conjecture that $\mathcal{C} = \mathcal{P}$ (within the class of groups with $|G| \leq 2^{\mathfrak{c}}$). This follows from our main conjecture: every group in $\mathcal{C}$ is either finite or abelian.

Problem

Describe the potentially dense subsets of a free countable non-abelian group F. Is it true that every $\mathfrak{Z}$-dense subset of F is potentially dense?

Recall that $\mathfrak{M}_F = \mathfrak{Z}_F = \mathfrak{C}_F$, so the $\mathfrak{Z}$-dense (i.e., $\mathfrak{M}$-dense) subsets of F are the subsets that are not contained in a finite union of cosets of cyclic subgroups of F.

Problem

Describe the potentially dense subsets of the Heisenberg group $\mathbb{H}_{\mathbb{Q}}$. Is it true that every $\mathfrak{Z}$-dense subset of $\mathbb{H}_{\mathbb{Q}}$ is potentially dense?

Recall that $\mathfrak{Z}_{\mathbb{H}_{\mathbb{Q}}} = \mathfrak{M}_{\mathbb{H}_{\mathbb{Q}}} = \mathfrak{C}_{\mathbb{H}_{\mathbb{Q}}} \vee \mathrm{cofin}_{\mathbb{H}_{\mathbb{Q}}}$.

Further applications of Markov-Zariski topology

Every proper closed subgroup H of a connected group G has index $\geq \mathfrak{c}$ as the homogeneous space G/H is connected and $T_{3.5}$.

Therefore, if a group G admits a connected group topology, then

(M) *all proper unconditionally closed subgroups of G have index at least $\mathfrak{c}$.*

Problem (Markov, Mat. Sbornik 1946)

Does every group G satisfying (M) admit a connected group topology?

Pestov and Remus gave non-abelian counter-examples (the latter proposed the permutation group $G = Sym(X)$ over a set X with $|X| > \mathfrak{c}$).

Theorem (Shakhmatov, DD – Adv. Math. 2016)

An abelian group G admits a connected group topology iff G satisfies (M).

von Neumann kernel and minimal almost periodicity

A *character* of a topological abelian group G is a continuous homomorphism $G \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$.
The *von Neumann kernel* $n(G)$ of G is the subgroup of all points of G where all characters of G vanish. The group G is called

(a) *minimally almost periodic* (briefly, MinAP), if every character $G \to \mathbb{T}$ is trivial (i.e., precisely when $n(G) = G$)

(b) *maximally almost periodic* (briefly, MAP), if the characters $G \to \mathbb{T}$ separate the points of G (i.e., when $n(G) = \{0\}$).

(In the non-abelian context $\prod_n \mathbb{U}(n)$ is used in place of $\mathbb{T}$.)
Obviously, $G/n(G)$ is MAP. The ingredient AP (almost periodic) comes from the name of the uniformly continuous functions $f : G \to \mathbb{C}$ that factor through $G \to G/n(G)$, largely studied in analysis and elsewhere. (So, MAP groups have sufficiently many AP functions to separate the points of G while the only AP functions of a MinAP groups are the constants.)
These notions and results are due to John von Neumann.

The first known examples of MinAP groups came from Analysis, the spaces $\ell_p, 0 < p < 1$ (any vector topological spaces without non-trivial continuous functionals).
Nienhuys 1971 built a monothetic MinAP group G (monothetic means that G has a dense copy of $\mathbb{Z}$).
These examples are connected and notoriously non-easy.
The first explicit and also quite simple example of a countable MinAP group was given by Prodanov in 1980.
In 1983, Ajtai, Havas and Komlós provided MinAP topologies on $\mathbb{Z}$ and some countably infinite direct sums of simple cyclic groups, deducing that every abelian group has a non-MAP topology.
In his 1984 Z.Bl.-review to their paper Protasov asked whether *every infinite abelian group admits a MinAP group topology*.
Remus noticed in 1989 that $G = \mathbb{Z}(2) \times \mathbb{Z}(3)^\omega$ does not admit any MinAP topology.
Motivated by this example, Comfort modified the original Protasov's question to the following Comfort-Protasov-Remus Problem (excluding completely the bounded group):

Problem 2 [Question 521, Open Problems in Topology 1, 1990]

Does every unbounded Abelian group admit a MinAP topology? What about the countable case?

Zelenyuk and Protasov introduced in 1990 a new technique for building MinAP topologies on countable groups, using T-sequences. Applying this method, Gabriyelyan obtained a description of the bounded abelian groups admitting a MinAP topology. He also proved that all countable unbounded groups admit a MinAP topology, resolving the second part of Problem 2. He obtained these results as particular cases when trying to resolve the more general question of describing all subgroups H of a given abelian group G such that there exists a Hausdorff group topology τ on G with $n(G,\tau) = H$. This justifies the following definition:

Definition ("realizing von Neumann's kernel")

Let H be a subgroup of an abelian group G. We say that H is a *potential von Neumann kernel* of G, if there exists a Hausdorff group topology τ on G such that $n(G,\tau) = H$.

In these terms the above "realization problem" sounds as follows:

Problem 3 [Gabriyelyan 2009]

Describe all potential von Neuman kernels of an abelian group G.

Clearly, an abelian group G admits a MinAP topology if and only if G is potential von Neumann kernel of itself.
This problem was resolved first for "small" subgroups H (i.e., either bounded or countable):

Theorem (Gabriyelyan Topology Appl. 2014, Proc. AMS 2015)

A subgroup H of an abelian group G is a potential von Neumann kernel of G if one of the following conditions holds:

(a) *G is unbounded and H is either bounded or countable;*

(b) *G is bounded and contains $\bigoplus_\omega \mathbb{Z}(k)$, where $k = o(H)$.*

Corollary (Gabriyelyan Topology Appl. 2014, Proc. AMS 2015)

An abelian group G admits a MinAP topology if G is unbounded countable, or bounded and contains $\bigoplus_\omega \mathbb{Z}(k)$, where $k = o(G)$.

The origin of Problem 3

In analogy ("duality") to the obvious fact that $G/n(G)$ is MAP (i.e., $n(G/n(G)) = 0$), one may expect that the von Neumann kernel $n(G)$ is necessarily MinAP (i.e., $n(n(G)) = n(G)$).

Problem 4 [Gábor Lukásc, October 2004]

Is the subgroup $n(G)$ of a topological abelian group always MinAP?

Inspired by an implicit hint in Milan, Tonolo, DD [J.Pure Appl. Algebra 2005]), Lukásc [2006] built examples of group topologies on $G = \mathbb{Z}(p^\infty)$ having finite but non-trivial $n(G)$, so clearly $n(n(G)) = 0 \neq n(G)$. He asked for a description of the abelian groups that admit a group topology τ such that $n(G, \tau) \neq 0$ is finite. Partial results were obtained by Nguyen [2009]. The final solution seems somewhat unexpectedly simple and elegant:

Theorem (Gabriyelyan 2009)

An abelian group G admits a finite non-trivial subgroup that is potential von Neumann kernel iff G has finite non-trivial subgroups.

The following easy lemma is helpful for finding a necessary condition that all potential von Neumann kernels must satisfy.

Lemma (1)

The von Neumann kernel of a topological group G is contained in every open subgroup of G and contains every minimally almost periodic subgroup of G.

Proof.

If H is an open subgroup of G, then G/H is discrete, so it is maximally almost periodic. Since the characters of G/H separate points of G/H, we get $n(G) \subseteq H$. The second assertion is obvious. □

Corollary

If H is an open MinAP subgroup of a topological abelian group G, then $H = n(G)$.

This will be our way of proving that a given subgroup H of a group G is a potential von Neumann kernel.

The $\mathfrak{Z}_G$-connected component $c_{\mathfrak{Z}}(G)$ of an abelian group G

Call an abelian group G bounded, if $mG = 0$ for some $m > 0$, unbounded otherwise. Let $o(G)$ be the smallest $m > 0$ with $mG = 0$, if G is bounded. Otherwise, let $o(G) = 0$.
Following Givens and Kunen, let $eo(G)$ be the least $m > 0$ such that mG is finite, in case G is a bounded abelian group. Otherwise, let $eo(G) = 0$. If $o(G) > 0$, then $eo(G)|o(G)$.

Theorem (Shakhmatov, DD 2010)

The connected component $c_{\mathfrak{Z}}(G)$ of $(G, \mathfrak{Z}_G)$ is a closed finite index subgroup. More precisely, $c_{\mathfrak{Z}}(G) = G[m]$, where $m = eo(G)$.

Consequently,
(a) $c_{\mathfrak{Z}}(G)$ coincides with the intersection of all (finitely many) $\mathfrak{Z}_G$-closed subgroups of finite index.
(b) $(G, \mathfrak{Z}_G)$ is connected iff $eo(G) = o(G)$ (i.e., mG is either infinite or $mG = \{0\}$ for any $m \in \omega$). In particular, $(G, \mathfrak{Z}_G)$ is connected if G is unbounded.
(Leading example: $c_{\mathfrak{Z}}(\mathbb{Z}(3) \oplus \mathbb{Z}(2)^{\omega}) = \{0\} \oplus \mathbb{Z}(2)^{\omega} = G[2]$.)

Necessary conditions for the existence of a MinAP topology

Lemma (Necessary condition for potential von Neumann kernels)

All potential von Neumann kernels H of an abelian group G are contained in $c_{\mathfrak{Z}}(G)$.

Proof. Indeed, if H is a potential von Neumann kernel witnessed by some Hausdorff group topology τ with $H = n(G, \tau)$, then $c_{\mathfrak{Z}}(G)$ being un unconditionally closed subgroup of G of finite index is τ-open, so $H \leq c_{\mathfrak{Z}}(G)$ by Lemma (1). □

By taking $H = G$ in the above lemma, one obtains the following necessary condition for the existence of a MinAP topology on arbitrary abelian groups.

Corollary

If an abelian group G admits a MinAP topology, then G is $\mathfrak{Z}_G$-connected.

We show that surprisingly, this quite simple and weak necessary conditions is also sufficient for the existence of a MinAP topology.

Theorem (D. Shakhmatov, DD 2014)

For an abelian group an abelian group G, the following are equivalent:

(a) *G admits a MinAP group topology;*

(b) *G is $\mathfrak{Z}$-connected;*

(c) *all proper unconditionally closed subgroups of G have infinite index;*

(d) *for every $m \in \mathbb{N}$, either $mG = \{0\}$ or $|mG| \geq \omega$.*

Since unbounded groups are $\mathfrak{Z}$-connected, we obtain as immediate corollary a complete solution of Problem 2:

Corollary

Every unbounded abelian group admits a MinAP topology.

As another corollary we obtain also complete solution of Problem 3:

Corollary (D. Shakhmatov, DD 2014)

A subgroup H of an abelian group G is a potential von Neumann kernel iff $H \leq c_3(G)$.

Proof. The necessity was proved above. To prove the sufficiency, assume that $H \subseteq c_3(G)$ and consider two cases.

Case 1. H is bounded. If G is unbounded, then H is a potential potential von Neumann kernel by Gabiyelyan's theorem.

Suppose now that G itself is bounded. Since $H \subseteq c_3(G)$ by our assumption, and $c_3(G) = G[m]$ (with $m = eo(G)$), so G contains $\bigoplus_\omega \mathbb{Z}(m)$ (Shakh.DD [2010]). As $mH = 0$, $k = o(H)$ divides m, so G contains $\bigoplus_\omega \mathbb{Z}(k)$. Now H is a potential von Neumann kernel of G again by Gabiyelyan's theorem.

Case 2. H is unbounded. We apply the Main theorem to find a MinAP topology τ on H. Extend τ to a Hausdorff group topology τ^* on G by taking as a base of τ^* all translates $g + U$, where $g \in G$ and $U \neq \emptyset$ is a τ-open subset of H. Since H is τ^*-open and (H, τ) is minimally almost periodic, one has $H = n(G, \tau^*)$. Therefore, H is a potential von Neumann kernel of G.

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