Dipartimento
di Matematica
Università di Pisa

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compensation in partial differential equations

Adolfo Arroyo Rabasa — adolfo.rabasa@unipi.it — www.arroyorabasa.com

IN SUPREMÆ DIGNITATIS
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overview

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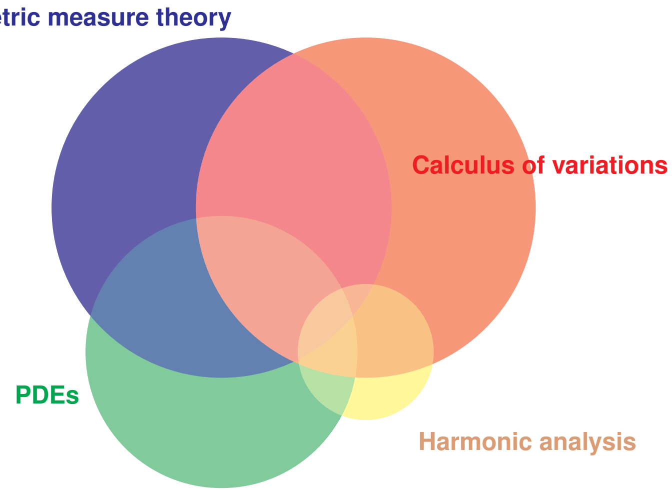
THE POSITIONS

2x PhD Positions (Q3-2025) $\rightsquigarrow$ **ERC group** (other postdocs, PhDs)

- regularity and rigidity of non-linear PDEs
- geometry of measures satisfying a PDE
- variational phenomena associated with PDE-constrains

AREAS OF STUDY



differential inclusions

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DIFFERENTIAL INCLUSIONS

- $\Omega \subset \mathbb{R}^n$ open and bounded (Lipschitz domain)
- $M \subset \mathbb{R}^{m\times n}$ smooth compact manifold

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$$Du(x) \in M$$

Can M...

Q. improve the *regularity* of M?

Q. affect the structure of Du?

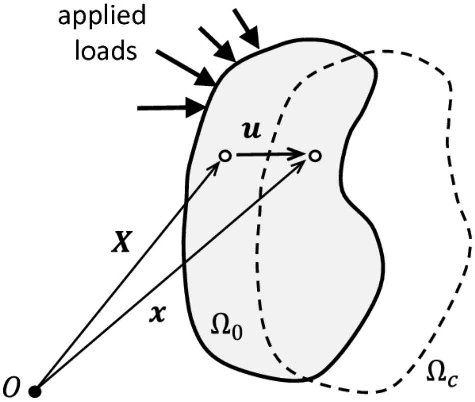
MOTIVATION (NONLINEAR ELASTICITY THEORY)

Displacement

$$u : \Omega_0 \subset \mathbb{R}^3 \to \Omega_c$$

Strain Displacement

$$Du : \Omega_0 \to \mathbb{R}^{3\times 3}$$



MODEL

Bodies minimize the elastic energy

$$\mathcal{E}[u] = \int_{\Omega_0} W(Du), \qquad u(\partial\Omega_0) = \partial\Omega_c$$

where $W : \mathbb{R}^{3\times 3} \to \mathbb{R}$ satisfies

$$W(AQB) = W(Q) \qquad \forall A, B \in \mathrm{SO}(3)$$

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$W(F) = \mathrm{dist}(F, \mathrm{SO}(3))^p$

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RIGIDITY RESULTS

Theorem (Liouville's theorem - Exact rigidity)

If a smooth map $u : \Omega \to \mathbb{R}^n$ satisfies $Du \in \mathrm{SO}(n)$,

$$u(x) = Rx + b\,, \qquad R \in \mathrm{SO}(n)\,,\ b \in \mathbb{R}^n$$

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Theorem (Friesecke–James–Müller - Quantitative rigidity)

If $u \in W^{1,p}(\Omega, \mathbb{R}^n)$, *then*

$$\min_{R \in \mathrm{SO}(n)} \int_\Omega |Du - R|^p \leqslant C(\Omega) \int_\Omega \mathrm{dist}(Du, \mathrm{SO}(n))^p$$

PHD PROJECT I

Generalize Friesecke–James–Müller to other manifolds (or Lie Groups):

Characterize (geometrically) those manifolds *M* satisfying

- A Liouville property (a priori regularity)
- Quantitative estability (regularity and rigidity)

Tools

- Sobolev Spaces and Partidal Differential Operators
- Regularity Theory
- (Basic) Riemannian Geometry
- Singular Integrals (Harmonic Analysis)

structure of gradient concentrations

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GRADIENT CONCENTRATIONS

Failure of L^1 convergence:

- fine oscillations
- concentrations of mass

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• $(Du_j) : \Omega \to \mathbb{R}^{m\times n}$ sequence converging weakly in the sense of measures:

$$\int_\Omega \phi \cdot Du_j \to \int_\Omega \phi \cdot d\mu \qquad \forall \phi \in C_0(\Omega; \mathbb{R}^{m\times n})$$

Question. Can we describe how mass is concentrated along the sequence?

MOTIVATION (LINEAR GROWTH FUNCTIONALS)

Models in

- Geometry (Area functional)
- Materials Science (Plasticity)
- Conservation Laws (Fluids)

involve the understanding of gradient concentrations

Example

Minimization principle

$$\min_u \int_\Omega F(Du), \qquad F(A) \lesssim 1 + |A|$$

THE RANK-ONE THEOREM

Concentrations may generate measure singularities

Framework: Functions of bounded variation

$$BV(\Omega)^m = \{u : \Omega \to \mathbb{R}^m : Du \text{ is a measure}\}$$

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Theorem (Rank-one theorem - Alberti)

Let $E \subset \Omega$ be a negligible set. If $u \in BV(\Omega)^m$ is written in polar form as

$$Du = g|Du|$$

then

$$\operatorname{rank} g(x) = 1 \qquad |Du| \text{ almost everywhere on } E$$

PHD PROJECT II

Conjecture: Gradient concentrations are the superposition of concentrations with rank-one averages

Tools:

- Compensated Compactness (PDEs)
- Young Measures (Geometric Measure Theory)
- *BV* spaces and Theory of Differential Operators
- Singular Integrals (Harmonic Analysis)

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Bibliography

- G. Friesecke, R. D. James and S. Müller, A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity, *Comm. Pure Appl. Math.* **55** (2002)
- G. Alberti, Rank one property for derivatives of functions with bounded variation, *Proc. Roy. Soc. Edinburgh Sect. A*
- A. Arroyo-Rabasa, Characterization of generalized young measures generated by $\mathcal{A}$-free measures, *Arch. Ration. Mech. Anal.*

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