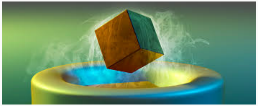
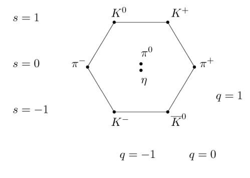
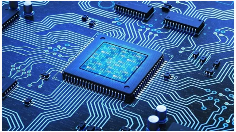
Math PhD days - Many-body Quantum Systems in Mathematical Physics

Chiara Boccato
Università di Pisa
January 21st 2025

Many-body Quantum Systems in Mathematical Physics







Quantum mechanics: central theory in physics, describing elementary particles, superconductors, quantum computers, ...

Challenging mathematical problems in functional analysis, partial differential equations, operator theory, statistical mechanics, ...
Very active current research topic.

Mathematical Physics: rigorous mathematical analysis of physically motivated problems. - Connections to the presentation of *C. Bonanno, P. Giulietti.*
Connections to Mathematical Analysis: *L. Forcella, J. Bellazzini, V. Georgiev.*

Many-Body Systems:

Small scale: we describe the world through **microscopic building blocks** (atoms, electrons, data, ...)

Large scale: we observe the **emergence of macroscopic patterns**, collective behavior (materials, superconductors, artificial intelligence, ...)

Giorgio Parisi, Nobel prize 2021:
"if you know how a single neuron behaves, that doesn't mean you understand how the brain behaves"

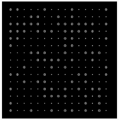
Small scale $\rightarrow$ *Large scale*: **new mathematics needed**!

Our Research: Mathematical Physics of Many-Body Interacting System

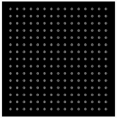
Many-body quantum mechanics

System of a large number of particles described microscopically by the Schrödinger equation.

Macroscopically we observe emergent phenomena, such as **phase transitions, universality, nonlinear effects.**



disorder



order

Curse of dimensionality: no exact analysis possible.
The emergent behaviour is *deeply affected by the interaction* among particles.

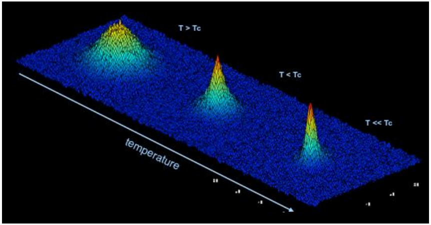
The challenge

Derive **effective theories** from first principles of quantum mechanics, describing the emergent physics in terms of *few degrees of freedom*.

Research Line:
Interacting Systems of Bosonic Particles

Emergence of Bose-Einstein condensation in gas of bosonic particles at very low temperatures.

1995: First experimental observation, awarded with the Nobel Prize in 2001[1]



Macroscopic manifestation of a quantum phenomenon, testing ground for entanglement and quantum correlations: a door to the quantum world. Connected to superfluidity, superconductivity, symmetry breaking.

Mathematical challenge: curse of dimensionality, perturbation theory fails

Objects of study: proof of Bose-Einstein condensation, study the **time evolution** and show the validity of effective theories, ...

[1] Anderson, Ensher, Matthews, Wieman, Cornell, *Science* **269** (1995)
Davis, Mewes, Andrews, van Druten, Durfee, Kurn, Ketterle. *Phys. Rev. Lett.* **75** (1995)

One-Body Quantum Mechanics

We consider one particle in $\mathbb{R}^3$.

We describe the **state of the system** through a wave function $\psi \in L^2(\mathbb{R}^3)$.

- $|\psi(\cdot)|^2$: probability density for the particle's position.

  Normalization: $\|\psi\|_2^2 = \int_{\mathbb{R}^3} |\psi(x)|^2 dx = 1$.
- In Fourier representation: $\widehat{\psi}(p) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ip\cdot x}\psi(x)dx$

  $|\widehat{\psi}(\cdot)|^2$: probability density for the particle's momentum.

Observable quantities are represented as self-adjoint operators A on $L^2(\mathbb{R}^3)$

- The energy is described by the operator, called Hamiltonian,

$$H = -\Delta + v$$

  where Δ is the laplacian and $v : \mathbb{R}^3 \to \mathbb{R}$ is a multiplication operator.

The **expectation value** of an observable A for a system in the state ψ is the inner product

$$\langle \psi, A\psi \rangle = \int \bar{\psi}(x)(A\psi)(x)dx$$

.

One-Body Quantum Mechanics

We are interested in

- The **ground state energy** is

$$E = \inf_{\substack{\psi \in L^2(\mathbb{R}^3), \\ \|\psi\|_2 = 1}} \langle \psi, H\psi \rangle$$

- The **ground state vector** solves the eigenvalue problem (time independent Schrödinger equation)

$$H\psi = E\psi$$

- The **spectrum** $\sigma(H)$: excitation energies
- The **dynamics**: given the initial state ψ_0, the time evolution ψ_t is described by the time dependent Schrödinger equation

$$i\partial_t \psi_t = H\psi_t$$

Example: Harmonic Oscillator.

$$H = -\Delta + \frac{\omega^2}{2}x^2 \qquad \text{acting on } L^2(\mathbb{R})$$

The **ground state energy** is

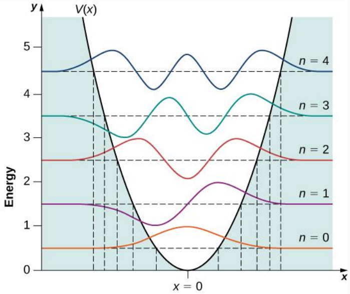
$$E = \frac{\omega}{2}$$

The **ground state vector** is

$$\psi(x) = \frac{\omega^2}{4\pi^2} e^{-\frac{\omega}{4}x^2}$$

The **spectrum** is $\sigma(H) = \{E_n\}$, where

$$E_n = \left(n + \frac{1}{2}\right)\omega, \quad \text{with } n \in \mathbb{N}$$



Consider N noninteracting particles in $\mathbb{R}^3$ described by

$$H_N = \sum_{i=1}^{N} \big(-\Delta_{x_i} + v_{ext}(x_i) \big)$$

acting on $\psi \in \underbrace{L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3) \otimes \cdots \otimes L^2(\mathbb{R}^3)}_{N \text{ times}} \cong L^2(\mathbb{R}^{3N})$.

Indistinguishable particles: bosons and fermions

- bosons have permutation-symmetric wavefunctions: $\psi \in L^2_s(\mathbb{R}^{3N})$

$$\psi(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N) = \underbrace{(+1)}_{\text{bosons}} \psi(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_N)$$

- fermions have permutation-antisymmetric wavefunctions: $\psi \in L^2_a(\mathbb{R}^{3N})$

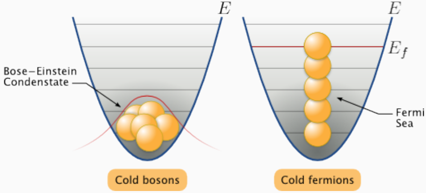
$$\psi(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N) = \underbrace{(-1)}_{\text{fermions}} \psi(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_N)$$

Many-body Quantum Mechanics

If we neglect interactions, wave functions are symmetrized or antisymmetrized products of functions

$$\psi = P_\pm\left(\varphi_{k_1} \otimes \varphi_{k_2} \otimes \cdots \otimes \varphi_{k_N}\right)$$

with $\varphi_k \in L^2(\mathbb{R}^3)$ appearing n_k times.



For fermions, $n_k \in \{0, 1\}$

For bosons, $n_k \in \{0, 1, 2, \ldots, N\}$

Bosons at low temperature exhibit perfect Bose-Einstein condensation[2]:

$$\psi(x_1, \ldots, x_N) = \varphi_0(x_1)\varphi_0(x_2)\ldots\varphi_0(x_N)$$

[2] Bose. *Z. Phys.* **26** (1924)
Einstein. *Sitzungsber. Preuss. Akad. Wiss.* (1924)

The interacting Bose gas

N **interacting bosonic** particles in $\mathbb{R}^3$

$$H_N = \sum_{i=1}^{N} \big(-\Delta_{x_i} + v_{ext}(x_i) \big) + \sum_{i<j}^{N} V(x_i - x_j)$$

acting on $\psi \in L^2_s(\mathbb{R}^{3N})$: symmetric tensor product $\Big(\underbrace{L^2(\mathbb{R}^3) \otimes \cdots \otimes L^2(\mathbb{R}^3)}_{N} \Big)_{\text{sym}}$

ψ is not factorized anymore!

$$\psi(x_1, \dots, x_N) \neq \varphi_0(x_1)\varphi_0(x_2)\dots\varphi_0(x_N)$$

Correlations

Interactions introduce **correlations:**
the many-body wave function ψ is far from a product (it is a linear combination of elementary tensors).

We need an efficient way to understand this.

We are interested in the following questions:

- Can we study the **dynamics** of a many-body system?
  How do nonlinear effects originate?

- Can we resolve the **spectrum** $\sigma(H_N)$ and the eigenfunctions?
  Can we prove condensation?

- Can we take the **large volume limit** (thermodynamic limit)?
  i.e. prove estimates that are uniform in the volume: this is crucial in statistical mechanics to understand phase transitions

- Can we include **temperature** effects?
  Can we compute the critical temperature in the interacting case?

The time-dependent Schrödinger equation

Cauchy problem: initial data $\psi_N \in L^2(\mathbb{R}^{3N})$,

Schrödinger equation $i\partial_t \psi_{N,t} = H_N \psi_{N,t}$.

Assume the initial data is a product state

$$\psi_N(x_1, x_2, \ldots, x_N) = \prod_{j=1}^{N} \varphi(x_i) \qquad \text{with} \quad \varphi \in L^2(\mathbb{R}^3)$$

Can we obtain its evolution $\psi_{N,t} = e^{-iH_N t}\psi_N$?

Due to the $V(x_i - x_j)$-terms in H_N the solution $\psi_{N,t}$ is not a product for any $t > 0$!

$$\psi_{N,t}(x_1, x_2, \ldots, x_N) \neq \prod_{j=1}^{N} \varphi_t(x_i) \qquad \text{with} \quad \varphi_t \in L^2(\mathbb{R}^3)$$

Nonlinear effective time evolution

We can however define an effective dynamics

nonlinear Schrödinger equation:	$i\partial_t \varphi_t = -\Delta\varphi_t + 8\pi\mathfrak{a}|\varphi_t|^2\varphi_t$
	$\varphi_0 = \varphi \in L^2(\mathbb{R}^3)$

so that

$$\lim_{N\to\infty} \left\| \psi_{N,t} - \sum_{k=0}^{N} \varphi_t^{\otimes(N-k)} \otimes_s u_k(t) \right\|_{L^2(\mathbb{R}^{3N})} = 0$$

for suitable functions $u_k(t) \in L^2(\mathbb{R}^{3k})$ easy to calculate.
The parameter $\mathfrak{a}$ is universal, i.e., independent on the microscopic shape of V.

The proof combines techiques from analysis of PDEs, functional analysis, quantum mechanics.

Emergent nonlinear behavior, while the original many-body problem is linear.
$\to$ blow-up, turbulence, solitons (stable travelling waves - impossible for the free Schrödinger equation because of dispersivity), ...

The spectrum of H_N

N bosons in a box $\Lambda = [-L/2, L/2]^3$

$$H_N = -\sum_{i=1}^{N} \Delta_{x_i} + \sum_{i<j}^{N} N^2 V\big(N(x_i - x_j)\big)$$

acting on $\psi \in L^2_s(\Lambda^N)$.
(We have translation invariance and a conserved momentum.)

The spectrum of $H_N - E_N$ is given by

$$\sum_{p \in \Lambda_+^*} n_p \sqrt{|p|^4 + 16\pi \mathfrak{a} p^2} + \mathcal{O}(N^{-1/4})$$

with $n_p \in \mathbb{N}$ and $n_p \neq 0$ for finitely many $p \in \Lambda_+^*$ only (n_p is the number of excited states with momentum p).

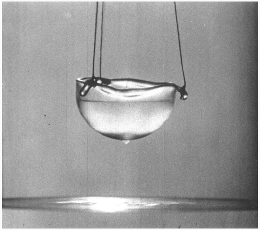
The excitation spectrum

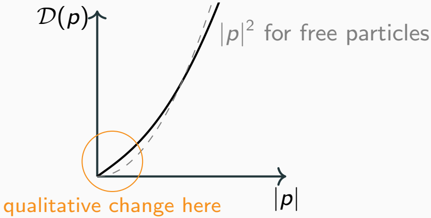
The dispersion relation of excitations

$$\mathcal{D}(p) = \sqrt{|p|^4 + 16\pi\mathfrak{a}p^2} = \sqrt{16\pi\mathfrak{a}}|p|\big(1 + \mathcal{O}(p^2)\big).$$

is linear for small momenta $\longrightarrow$ signature of superfluidity (frictionless flow)

Effect due to INTERACTIONS.



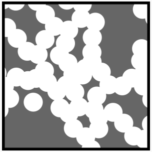


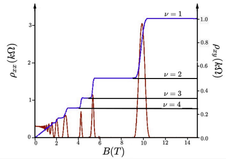
This is in agreement with the predictions of Bogoliubov and Landau but its proof requires a **new description of correlations**, to justify **universality** of the spectrum depending only on $\mathfrak{a}$.

- **Correlations in Bosonic Systems**

  Temperature effect in the Bose gas:
  $\rightarrow$ *find minimizers of a free energy functional*

  Many-body time evolution of bosonic systems:
  $\rightarrow$ *prove a norm approximation for the dynamics of excitations*



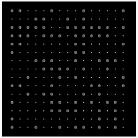


- **Quantum Systems with Disorder**

  Localization of ground states, non-ergodic behavior

- **Effective Description of Quantum Hall States**

  Dynamics of many-body fermionic systems in a magnetic field



- **Ising Model**

  Quasi-periodic disorder, universality of critical exponents

Grants and Collaborations

PRIN 2022 Grant "Interacting Quantum Systems: Topological Phenomena and Effective Theories"

Network of Collaborators: U Zurich, IST Austria (Vienna), VirginiaTech, U Bonn, SISSA Trieste, U La Sapienza, U Hagen, GSSI L'Aquila, U Milano, ...