Algebraic Geometry and Geometric Representation Theory

Andrea Di Lorenzo, Francesco Sala, and Mattia Talpo

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Moduli spaces

Definition

Moduli space M = a space (algebraic variety) whose points parametrize some fixed kind of objects, in a functorial way

i.e. if $X \to S$ is a family of objects parametrized by M, there is an associated classifying morphism

$$S \longrightarrow M\,, \quad s \longmapsto \text{the fiber of } X \to S \text{ over } S$$

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Examples

- For a fixed genus g, there is a moduli space M_g parametrizing smooth proper connected algebraic curves of genus g.
- If C is a smooth curve, there is a moduli space $M^{ss}_{r,d}(C)$ of (semi-stable) vector bundles on C of degree d and rank r.

The geometry of these spaces is usually very interesting and rich, and there are numerous connections to other areas of mathematics and physics.

Warning

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Equivalently, there rarely is a universal family $U \to M$ (i.e. that recovers any other family $X \to S$ as pullback along the classifying morphism $S \to M$).

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Example: quotient stacks

If G is an algebraic group acting on a scheme X, there is always a quotient stack $[X/G]$, for which the projection $X \to [X/G]$ is a G-principal bundle.

The points of $[X/G]$ correspond to orbits of the action, and the automorphism group is the isotropy group of any point of the orbit.

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"Nice" moduli stacks exist far more often than fine moduli spaces.

For example, there are algebraic stacks $\mathcal{M}_g$ and $\mathcal{M}_{r,d}^{ss}(C)$ parametrizing smooth proper connected genus g curves and semi-stable vector bundles of degree d and rank r on C.

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Base field: $\mathbb{C}$.

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- Geometry of families of elliptic curves: we can always* find a filling (in a unique way) of a family of elliptic curves over a punctured disk, **if we allow nodal curves**!

Here is one of the major achievements of the last century for what concerns moduli of curves (two Fields medals here!).

Theorem (Deligne–Mumford, 1969)

The moduli stack $\mathcal{M}_{g,n}$ can be compactified by adding nodal algebraic curves having finitely many automorphisms (technically, stable curves).

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Recently, Giovanni Inchiostro and Andrea have been exploring new directions for tackling this problem.

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Here are a couple of recent results on this topic that Andrea and his collaborators managed to prove:

Chow ring of $\overline{\mathcal{M}}_{2,1}$ (D.L.–Pernice–Vistoli, 2022)

$$\mathrm{CH}^*(\overline{\mathcal{M}}_{2,1}) \simeq \mathbb{Z}[\lambda_1, \psi_1, \vartheta_1, \lambda_2, \vartheta_2] / (\alpha_{2,1}, \alpha_{2,2}, \alpha_{2,3}, \beta_{3,1}, \beta_{3,2}, \beta_{3,3}, \beta_{3,4}).$$

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Fun fact: the non-triviality of the Brauer group is related to the famous **27 lines on a smooth cubic surface**. It was quite surprising to discover this!

Can we compute further invariants in higher genus?

“Enhanced” version of algebraic geometry.

Objects are log schemes = a variety X + a sheaf of monoids M with $\alpha\colon M \to (\mathcal{O}_X, \cdot)$.

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Prototypical examples

Pairs (X, D) with X smooth and D a SNC divisor (e.g. $\mathbb{A}^n$ or $\mathbb{P}^n$ with the coordinate hyperplanes), toric varieties.

The extra data typically keeps track of either a (partial) compactification (e.g. $X \setminus D \subseteq X$), or of a family of which the space is a (typically singular) fiber.

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- It is "combinatorial" in nature, building blocks are affine toric varieties $\operatorname{Spec} k[P]$.
- There are interesting and fruitful connections to tropical geometry (i.e. "piecewise linear" algebraic geometry).

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Equipping objects with log structures allows for better control over degenerations. This has applications to (for example) enumerative geometry and mirror symmetry.

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I also work with parabolic bundles, a notion of decorated vector bundles that naturally live on log schemes.

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There are moduli spaces and stacks for semi-stable parabolic vector bundles, and their geometry hasn't been studied very much, especially in the singular (but log smooth) case.

Some understanding of these should be useful in studying degenerations of moduli spaces of vector bundles on smooth varieties.

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Motivating Example: the topology of Hilbert schemes of points on S

Symmetric product of a smooth surface

S = smooth (quasi-)projective complex surface. For any $n \in \mathbb{N}$, define

$$\mathsf{Sym}^n(S) := \left(S \times \cdots \times S\right) \Big/ \mathfrak{S}_n = \text{moduli space of } \textit{unordered } n\textit{-tuples of points of } S$$

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- $\mathsf{Hilb}^n(S)$ is a smooth complex variety
- $\mathsf{Hilb}^n(S)$ = moduli space of *ideal sheaves* $\mathcal{J}_Z \subset \mathcal{O}_S$ of 0-dimensional subschemes $Z \subset S$ of length n

Set $b_i(-) := \dim H^i(-;\mathbb{Q})$ = i-th Betti number

Göttsche-Soegel: topology of $\mathsf{Hilb}^n(S)$

$$\sum_{n\in\mathbb{N}}\sum_{i\in\mathbb{N}} b_i(\mathsf{Hilb}^n(S))\, t^i\, q^n = \prod_{m=1}^{+\infty} \frac{(1+t^{2m-1}q^m)^{b_1(S)}(1+t^{2m+1}q^m)^{b_3(S)}}{(1-t^{2m-2}q^m)^{b_0(S)}(1-t^{2m}q^m)^{b_2(S)}(1-t^{2m+2}q^m)^{b_4(S)}}$$

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Nakajima: $\mathsf{Hilb}^n(S)$ vs. Heisenberg algebra

$\exists$ an action of Heis_S on $\mathbb{V}_0(S) := \bigoplus_n H^*(\mathsf{Hilb}^n(S);\mathbb{Q})$ such that $\mathbb{V}_0(S) \simeq$ Fock space

$\Longrightarrow$ we can describe explicitly all cohomology classes in $\mathbb{V}_0(S)$

Set $b_i(-) := \dim H^i(-;\mathbb{Q}) = i$-th Betti number

Göttsche-Soegel: topology of $\mathrm{Hilb}^n(S)$

$$\sum_{n\in\mathbb{N}}\sum_{i\in\mathbb{N}} b_i(\mathrm{Hilb}^n(S))\, t^i q^n = \prod_{m=1}^{+\infty} \frac{(1+t^{2m-1}q^m)^{b_1(S)}(1+t^{2m+1}q^m)^{b_3(S)}}{(1-t^{2m-2}q^m)^{b_0(S)}(1-t^{2m}q^m)^{b_2(S)}(1-t^{2m+2}q^m)^{b_4(S)}}$$

= character of the Fock space of the (super-)Heisenberg algebra Heis_S associated to $H^*(S;\mathbb{Q})$

Nakajima: $\mathrm{Hilb}^n(S)$ vs. Heisenberg algebra

$\exists$ an action of Heis_S on $\mathbb{V}_0(S) := \bigoplus_n H^*(\mathrm{Hilb}^n(S);\mathbb{Q})$ such that $\mathbb{V}_0(S) \simeq$ Fock space

$\Longrightarrow$ we can describe explicitly all cohomology classes in $\mathbb{V}_0(S)$

Generalization

- *Schiffmann-Vasserot*, *Neguţ*: from cohomology to K-theory
  Heis_S is replaced by the Elliptic Hall algebra (depending on $K^0(S)$)
- *Schiffmann-Vasserot*, *Soibelman* & co, *Neguţ*, etc: from $\mathrm{Hilb}^n(S)$ to other moduli spaces (quiver varieties, instanton moduli spaces, Gieseker moduli spaces, etc)

Warning

- Not only *topology of moduli spaces* $\Longleftrightarrow$ *representations of interesting algebras* (e.g., vertex algebras, quantum groups, etc.)
- but also *topology of moduli stacks* $\Longrightarrow$ *geometric realizations such algebras* (theory of Cohomological, K-theoretical, categorical Hall algebras)

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Examples of COHAs

(1) *Schiffmann-Vasserot*: $K_0(\mathbf{Coh}_{0\text{-}\dim}(\mathbb{C}^2))$ is endowed with a *Hall* multiplication s. t.

$$\mathbf{KHA}_{0\text{-}\dim}(\mathbb{C}^2) := (K_0(\mathbf{Coh}_{0\text{-}\dim}(\mathbb{C}^2)), *) \simeq \text{the elliptic Hall algebra}$$

(2) *Kapranov-Vasserot*: $\exists\, \mathbf{KHA}_{\dim\leqslant k}(S), \mathbf{COHA}_{\dim\leqslant k}(S)$ for any smooth surface S

(3) *Porta-S.*: $\exists$ *categorical Hall algebra* on $\mathsf{D}^b(\mathsf{Coh}(\mathbb{R}\mathbf{Coh}_{\dim\leqslant k}(S)))$

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Why is the theory of COHAs important?

- (1) & (2) $\Longrightarrow$ Proofs of conjectures in Enumerative Geometry (e.g. Alday-Gaiotto-Tachikawa conj., P=W conj., Okounkov's conj.)
- (3) $\Longrightarrow$ foundation for the theory of Categorical Enumerative Geometry due to *Pădurariu* and *Toda*.

- Andrea Di Lorenzo: Moduli stacks of curves and their invariants
- Marco Franciosi, Rita Pardini: birational geometry, moduli spaces of surfaces
- Gregory Pearlstein: Hodge theory
- Tamás Szamuely: cohomology of varieties and arithmetic questions
- Francesco Sala: cohomological Hall algebras and geometric representation theory
- Mattia Talpo: Logarithmic and tropical algebraic geometry

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