Geometric topology

21th January 2025

Low–dimensional topology

- n-dimensional manifolds, with $n = 2, 3, 4$
- Knot theory

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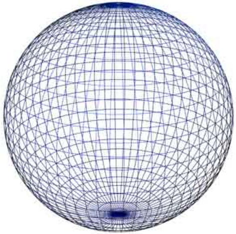
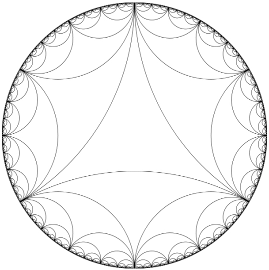
- Geometric structures

Geometric structures

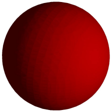
Spaces of constant curvature

- Euclidean space $\mathbb{R}^n$ $(K = 0)$
- Spherical space S^n $(K = 1)$
- Hyperbolic space $\mathbb{H}^n$ $(K = -1)$

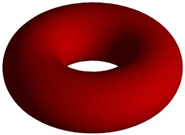




Compact surfaces



$g = 0$



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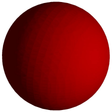

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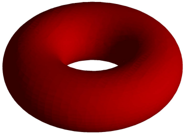

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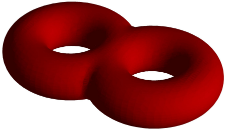

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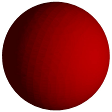
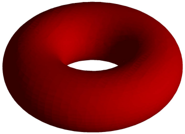


$g = 0$



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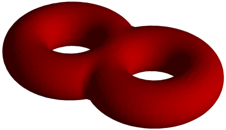
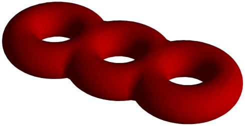


$g = 2$



$g = 3$

Uniformization

Theorem

Every compact surface admits a metric of constant curvature K, with:

- *if $g = 0$, then $K = 1$ (spherical case)*
- *if $g = 1$, then $K = 0$ (flat case)*
- *if $g = 2$, then $K = -1$ (hyperbolic case)*

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The Teichmüller space $\mathcal{T}(\Sigma)$ of a surface Σ is the space of metrics of constant curvature on Σ.

The hyperbolic case

If Σ is hyperbolic, then

$$\Sigma = \mathbb{H}^2/\rho(\pi_1(\Sigma))$$

where

$$\rho\colon \pi_1(\Sigma) \to \mathrm{Isom}(\mathbb{H}^2) = PSL(2, \mathbb{R})$$

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Geometric structures $\Longleftrightarrow$ Representations

Higher Teichmüller Theory (Andrea Tamburelli)

Goal: Study geometric structures on surfaces (Higgs bundles, space-time structures on $\Sigma \times \mathbb{R}$, ...) via representations

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All these theorems were proved by the Field Medalist Grigory Perelman.

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Let $n \geq 4$. Which n-dimensional manifolds support a hyperbolic structure?

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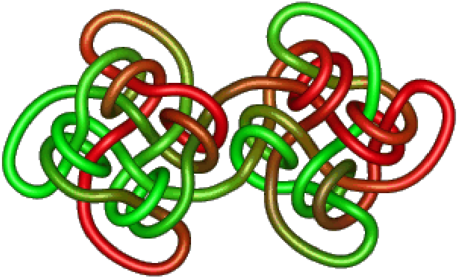

Knot Theory (Paolo Lisca)

- Classical techniques (triangulations, geometrization,. . .)
- Recent invariants (Osvath-Szabo invariants, Heegaard-Floer Homology,. . .).

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A manifold M is aspherical if $\pi_n(M) = 0$ for every $n \geq 2$ or, equivalently, if the universal covering $\widetilde{M}$ of M is contractible.

Other invariants of manifolds (Frigerio)

Conjecture (Gromov)

Let M be a compact aspherical manifold with vanishing simplicial volume. Then $\chi(M) = 0$.