Tame functors from posets to chain complexes

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Topology of Arrangements with an Eye to Applications
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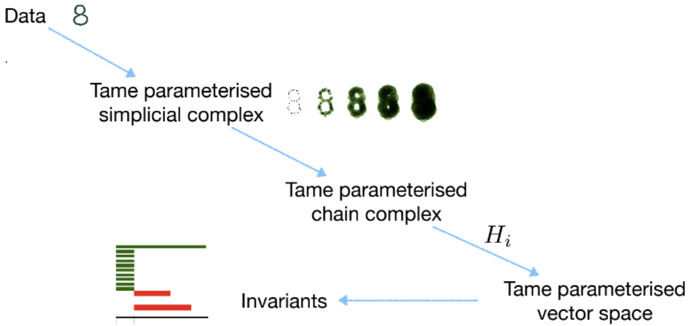
Persistent homology

Multiscale data analysis via parametrized functors $[0, \infty) \to \mathcal{C}$:



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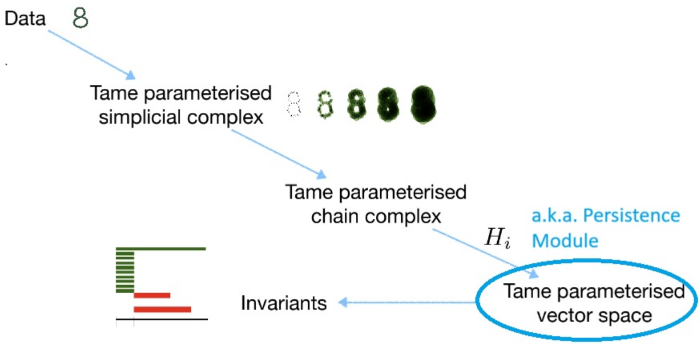

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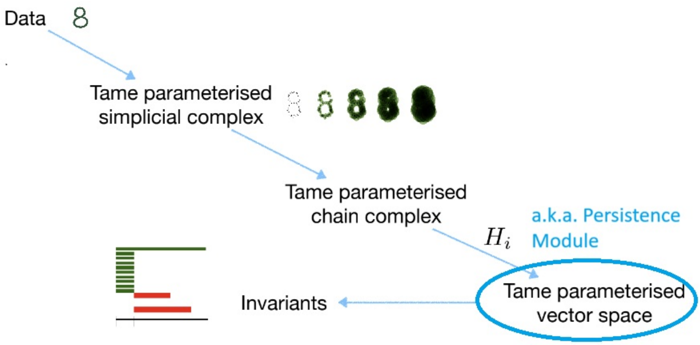


Key ingredients:

- applying homology to land in **vec**
- parametrization by a linear poset category like $[0, \infty) \subseteq \mathbb{R}$
- tameness

Reason for these choices

Gabriel's Theorem (1972)

Let $\mathcal{Q}$ be a finite connected poset category and let **vec** the category of finite dimensional vector spaces over a field $\mathbb{F}$. Then, there are finitely many isomorphism classes of indecomposable functors in $\mathrm{Fun}(\mathcal{Q}, \mathbf{vec})$ if and only if $\mathcal{Q}$ is of Dynkin type.

Structure theorem

For the poset is $[n] = \{0 < 1 < \cdots < n\}$, every functor in $\mathrm{Fun}([n], \mathbf{vec})$ uniquely decomposes as a direct sum of functors of the form

$$\begin{matrix} 0 & & & s & & & t & & & n \\ 0 \longrightarrow & \cdots 0 & \longrightarrow & \mathbb{F} & \xrightarrow{\mathrm{id}} \cdots \xrightarrow{\mathrm{id}} & & \mathbb{F} & \longrightarrow & 0 \cdots \longrightarrow & 0 \end{matrix}$$

Barcodes of parametrized vector spaces

Corollary

Left Kan extending along $[n] \subset [0, \infty)$, from

$$Y : [n] \to \mathbf{vec}$$

we obtain

$$X : [0, \infty) \to \mathbf{vec}$$

which can be identified with a set of intervals

$$\mathrm{Bar}(X) = \{[s_i, t_i)\}_i$$

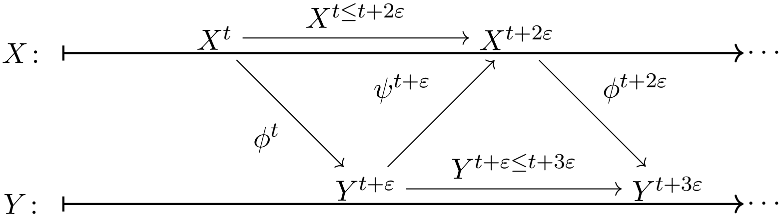
called the **barcode** of X.

The isometry theorem of Persistence

The **interleaving distance** between $X, Y : [0, \infty) \to \mathbf{vec}$ is defined as

$$d_I(X,Y) := \inf\{\varepsilon > 0 \mid \exists\ \varepsilon\text{-interleaving between } X \text{ and } Y\},$$

where an **ε-interleaving** between X, Y is a pair of morphisms $\phi\colon X \to Y^{\blacktriangle+\varepsilon}$ and $\psi\colon Y \to X^{\blacktriangle+\varepsilon}$ such that



commutes for all t.

The isometry theorem of Persistence

The **bottleneck distance** between two barcodes Bar and Bar′ is defined as

$$d_B(\mathrm{Bar}, \mathrm{Bar}') := \inf\{c(\mathcal{M}) \mid \mathcal{M} \subseteq \mathrm{Bar} \times \mathrm{Bar}' \text{ is a matching}\}.$$

The **cost** of a matching $\mathcal{M}$ between two barcodes $\mathrm{Bar}, \mathrm{Bar}'$ is defined as

$$c(\mathcal{M}) := \max\left\{ \sup_{(I,J)\in\mathcal{M}} c(I,J), \sup_{I\in \mathrm{Bar}\cup\mathrm{Bar}' \text{ unmatched}} c(I) \right\},$$

where for $I = [s,t)$ and $J = [s', t')$ we have

$$c(I,J) = \max\{|t - t'|, |s - s'|\}, \qquad c(I) = \frac{t-s}{2}.$$

Theorem (Lesnick)

The interleaving distance on tame functors $X, Y : [0,\infty) \to \mathbf{vec}$ is equal to the bottleneck distance on their corresponding barcodes $\mathrm{Bar}(X)$ and $\mathrm{Bar}(Y)$.

The stability theorem

Stability theorem for functions (Cohen-Steiner et al.)

Given two real-valued continuous functions f, g defined on the same topological space, let $X, Y : [0, \infty) \to \mathbf{vec}$ be parametrized tame vector spaces given by

$$X^t := H_i(f^{-1}(-\infty, t]) \quad Y^t := H_i(g^{-1}(-\infty, t])$$

$$d_I(X, Y) \leq \|f - g\|_\infty$$

So, for every $\varepsilon > 0$ and every real-valued smooth function f, if g is a PL function ε-approximating f, then their barcodes are ε-close in the bottleneck distance.

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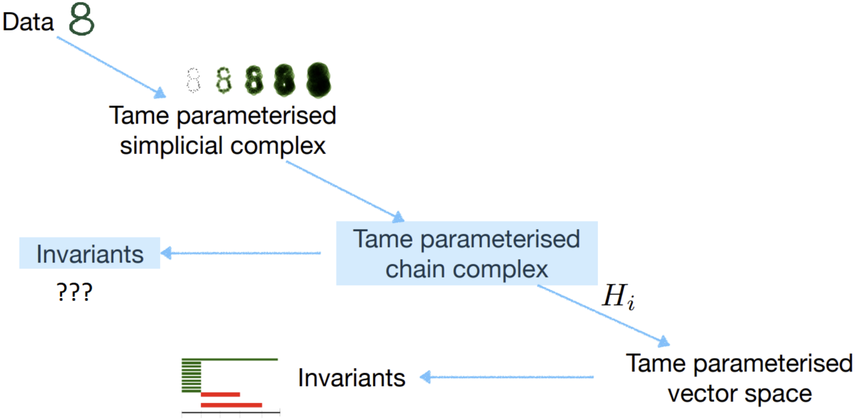
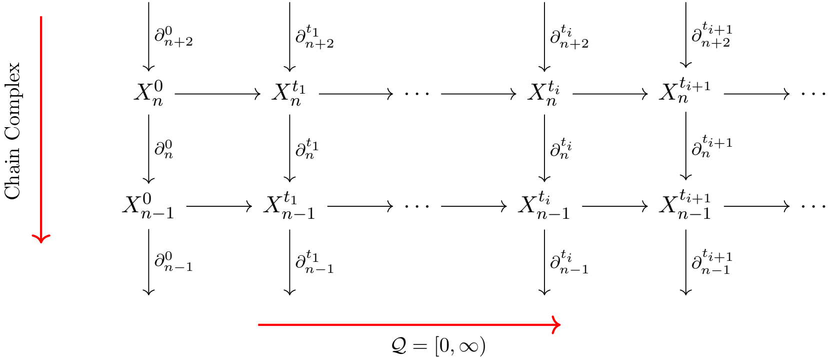
Goals for this talk

What if we:

- work directly with chain complexes in **ch**(**vec**)
- generalize to other poset categories $\mathcal{Q}$
- want to maintain tameness



- Functors $X : \mathcal{Q} \to \mathbf{ch}(\mathbf{vec})$ with $\mathcal{Q}$ a poset category and $\mathbf{ch}$ the category of non-negative chain complexes of vector spaces over a field $\mathbb{F}$



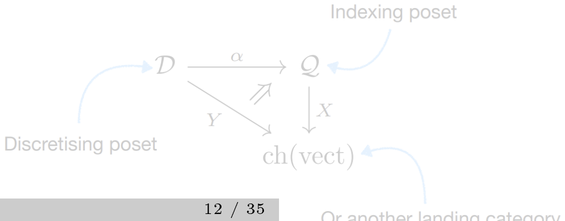
- Natural transformations between them as morphisms.

Tameness

Idea: $X \in \mathrm{Fun}(\mathcal{Q}, \mathbf{ch}(\mathbf{vec}))$ is **tame** if non-trivial in finitely many degrees + there is a finite poset $\mathcal{D} \subset \mathcal{Q}$ s.t. transition morphisms $X^{s \leq t} \colon X^s \to X^t$ may fail to be isomorphisms only when $[s, t) \cap \mathcal{D} \neq \emptyset$

$$X^0 \longrightarrow X^{t_1} \longrightarrow \cdots \qquad \cdots \longrightarrow X^{t_n}$$

Formally: $X \colon \mathcal{Q} \to \mathbf{ch}(\mathbf{vec})$ is **tame** if X is isomorphic to the left Kan extension $\alpha^k Y$ along a poset map $\alpha \colon \mathcal{D} \to \mathcal{Q}$ with $\mathcal{D}$ finite, of a functor $Y \colon \mathcal{D} \to \mathbf{ch}(\mathbf{vec})$ non-trivial only at finitely many degrees:



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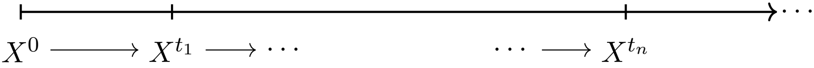

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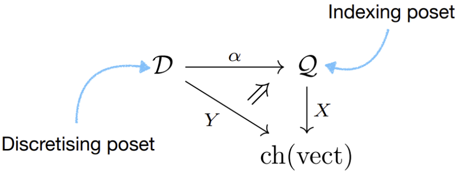

How to obtain invariants (barcodes)

$\text{Tame}(\mathcal{Q}, \mathbf{ch}(\mathbf{vec}))$ = full subcategory of tame functors in $\text{Fun}(\mathcal{Q}, \mathbf{ch}(\mathbf{vec}))$

Take an object X in $\text{Tame}(\mathcal{Q}, \mathbf{ch}(\mathbf{vec}))$	- - - →	Decompose X into "simple" indecomposables

We next examine 3 situations in which this is possible:

1. $\mathcal{Q} = [0, \infty)$ and X **filtered**
2. $\mathcal{Q} = [0, \infty)$ and X **factored**
3. $\dim \mathcal{Q} = 1$ and X **cofibrant**

A **filtered chain complex** is a tame parametrized chain complex $X\colon [0,\infty) \to \mathbf{ch}(\mathbf{vec}))$ whose transition morphisms $X^{s<t}\colon X^s \to X^t$ are monomorphisms.

$$X^0_\bullet \hookrightarrow X^{t_1}_\bullet \hookrightarrow \cdots \qquad \cdots \hookrightarrow X^{t_n}_\bullet \quad \cdots$$

Structure Theorem

Each filtered chain complex decomposes into a finite direct sum $X = \bigoplus_n \bigoplus_{[b,d)\in \mathrm{Bar}_n} I^n\,[b,d)$ where $I^n\,[b,d)$ is

$$\begin{array}{cccccccccccccc}
\cdots & & b-\varepsilon & & b & & \cdots & & d-\varepsilon & & d & & \cdots & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{F} & \longrightarrow & \cdots & n+1 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{F} & \longrightarrow & \cdots & \longrightarrow & \mathbb{F} & \longrightarrow & \mathbb{F} & \longrightarrow & \cdots & n
\end{array}$$

[Chachólski-Giunti-L. 2021: Invariants for tame parametrised chain complexes]

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 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
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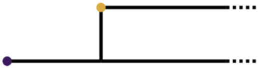
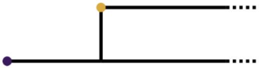
Application: A persistence algorithm



decomposition of tame filtered chain complexes	$\Longrightarrow$	decomposition of tame parametrized vector spaces

Application: A persistence algorithm



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$\downarrow$

For a parametrized simpicial complex $K^0 \subseteq K^1 \subseteq \cdots \subseteq K$, do:

- Find two simplices σ_i and σ_j s.t. $\sigma_j \in \partial\sigma_i$ & σ_i has the latest entrance time t among cofaces of σ_j & σ_j has the earliest entrance time s among faces of σ_i
- Append $[b, d)$ to the list of bars of degree $n = \dim \sigma_j$
- Split $I^n[b, d)$ from $\mathbf{ch}(K^\blacktriangle)$
- Repeat until possible, then append $[b, \infty)$ to the list of bars of degree $n = \dim \sigma$ for every remaining σ

[Chachólski-Giunti-Jin-L. 2023: Decomposing filtered chain complexes]

Factored chain complexes

A tame parametrized chain complex $X\colon [0,\infty) \to \mathbf{ch}(\mathbf{vec})$ is called **factored** if its transition morphisms $X^{s<t}\colon X^s \to X^t$ are epimorphisms.

$$X^0_\bullet \twoheadrightarrow X^{t_1}_\bullet \twoheadrightarrow \cdots \qquad\qquad \cdots \twoheadrightarrow X^{t_n}_\bullet$$

Structure Theorem $\quad \mathcal{I}^n \underbrace{[0,s,t)}_{\textbf{tagged interval}} =$

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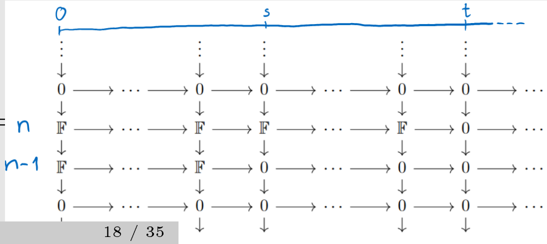


Isometry Theorem on factored chain complexes

A **tagged barcode** is a multiset of tagged intervals $[0, s, t) :=$ the interval $[0, t)$ + a distinguished point $s \in [0, t]$.

Given two tagged intervals $I = [0, s, t), J = [0, s', t')$, we set

- $c(I, J) := \max\{|t - t'|, |s - s'|\}$
- $c(I) := \frac{t}{2}$

yielding a well-defined **generalized bottleneck distance of tagged barcodes**.

Also, we have the generalized **interleaving distance** on parametrized chain complexes.

Isometry Theorem

For any factored chain complexes X, Y,

$$d_I(X, Y) = \max_{n \in \mathbb{N}} d_B(\mathrm{tBar}_n(X), \mathrm{tBar}_n(Y)).$$

Application: tagged barcodes for vector fields

gradient-like vector field v

↓

weighted based Morse Complex $(\mathrm{MC}_\bullet(v), \mathcal{B}_\bullet, \mathrm{w}_\bullet)$

↓

factored chain complex $X(\mathrm{MC}_\bullet(v)$

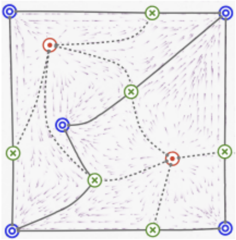
↓

tagged barcode $\coprod_{n,i}(n, [0, s_i, t_i))$

Gradient-like Morse Smale vector fields

Vector fields on a closed manifold M that are:

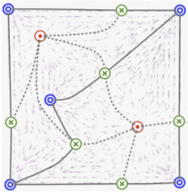
- **gradient-like**, i.e. without closed orbits
- **Morse-Smale**, i.e. with only hyperbolic singularities whose stable and unstable manifolds are transversal

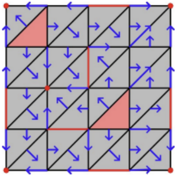


We also fix a Riemannian structure on M.

The **Morse complex** $MC_\bullet(v)$ is the chain complex where

- $MC_k(v)$ is the free $\mathbb{F}$-vector space generated by the singular points of index k of v
- $\partial : MC_k(v) \to MC_{k-1}(v)$ is defined by counting (mod 2) the flowlines between singular points of adjecent index.





$$\mathbb{F}^2 \xrightarrow{\partial_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}} \mathbb{F}^4 \xrightarrow{\partial_1 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}} \mathbb{F}^2$$

$(\mathbb{F} = \mathbb{Z}_2)$

Morse complex $\longrightarrow$ weighted based chain complex

We can turn $\mathrm{MC}_\bullet(v)$ into a **weighted based chain complex** by chosing for all k:

- as basis $\mathcal{B}_k$ of C_k the set of singular points of v of index k
- **weights** $\mathrm{w}_k\colon \mathcal{B}_k \times \mathcal{B}_{k-1} \to [0,\infty)$ given by the distance $d(a,b)$ for all $a \in \mathcal{B}_k$ and $b \in \mathcal{B}_{k-1}$.

If v is **in general position**, i.e. the distances between its singular points are pairwise distinct, then $\mathrm{MC}_\bullet(v)$ is **generic**, i.e. w_k is injective.

weighted based Morse complex $\longrightarrow$ factored chain complex

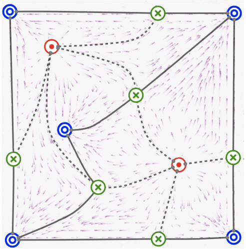
Given v a Morse-Smale vector field in general position, we define the factored chain complex $X = X(v)$ as follows:

- Set $t_0 = 0$ and let $X^{t_0} = \mathrm{MC}(v)$.
- Assume we have already defined a sequence of chain complexes $X^{t_0} \to X^{t_1} \to \cdots \to X^{t_i}$. We define
  - $t_{i+1} := t_i + d(a, b)$, where (a, b) is the pair of critical points with b in $\partial(a)$ and the smallest distance.
  - $X^{t_{i+1}} := X^{t_i}/\langle a, \partial a\rangle$ the result of simplifying X^{t_i} along the pair (a, b).
  - For $t_i < t < t_{i+1}$ we define $X^t := X^{t_i}$ and for $t_i \leq t \leq s < t_{i+1}$ we define $X^{s \leq t} = \mathbb{1}$.
- When we reach the point where all the differentials in the chain complex X^{t_r} are zero, then we stop the algorithm and define $X^t = X^{t_r}$ for all $t_r < t < \infty$ and $X^{t \leq s} = \mathbb{1}$ for all $t_r \leq t \leq s < \infty$.

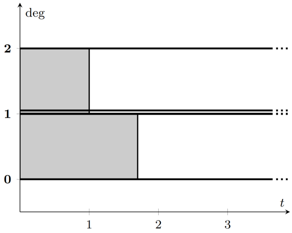
Stability

Theorem

The function $v \mapsto \mathrm{tBar}(X(v))$ is a continuous map if we endow the space of tagged barcodes with the intereleaving distance and the space of gradient-like Morse-Smale vector fields in general position with Whitney C^1-topology.



$\mapsto$



Proof Sketch

Let $v \in \mathfrak{X}_{gMS+}(M)$ and let $\varepsilon > 0$. Denote by $\operatorname{Sing}(v)$ the set of singular points of v.
Since Morse-Smale vector fields are **structurally stable** and their singular points are **locally structurally stable**, there exists a neighbourhoud $\mathcal{N}$ of v in $\mathfrak{X}_{gMS+}(M)$ such that for all $w \in \mathcal{N}$ we have

- $\forall p \in \operatorname{Sing}(v)\ \exists!\ q \in \operatorname{Sing}(w):\ \ d(p,q) \leq \varepsilon.$
- The vector fields v and w are topologically equivalent.

$\implies$ The algorithm applied to v and w simplifies pairs of corresponding singular points.

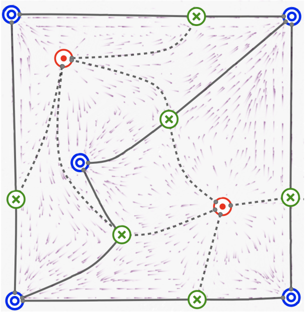
$\implies$ $d_I(X(v), X(w)) \leq n\varepsilon$, where $n = |\operatorname{Sing}(v)|$.

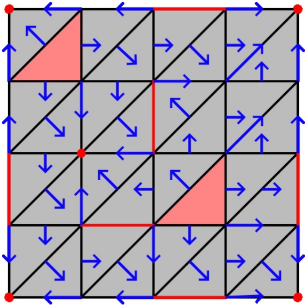
Combinatorial approximation

Theorem

Let M be an oriented Riemannian manifold and let $v \in \mathfrak{X}_{gMS+}(M)$. Let (M', ϕ, V) be a triangulation of v. Then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$d_I\left(X(\mathrm{MC}_\bullet(v)), X(\overline{MC}_\bullet(\Delta^n(V)))\right) < \varepsilon.$$





[Bannwart-L.: Tagged barcodes for the topological analysis of gradient-like vector fields]

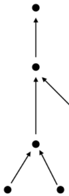
Issues with the general case

	$\mathrm{Tame}(\mathcal{Q}, \mathrm{ch}(\mathrm{vect})) \subset$	$\mathrm{Fun}(\mathcal{Q}, \mathrm{ch}(\mathrm{vect}))$
Abelian	✗	✓
Model	✗	✗

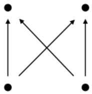
- problems with $\mathrm{Fun}(\mathcal{Q}, \mathbf{ch}(\mathbf{vec}))$ if the poset is not finite
- tameness is not preserved by taking finite limits
- some choices of $\mathcal{Q}$ solve the issues

One-dimensional posets

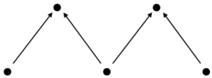
A poset $\mathcal{Q}$ has **dimension 1** if for every x in $\mathcal{Q}$, $\mathcal{Q} < x$ is empty or for any two noncomparable elements y, z in $\mathcal{Q} < x$, whose least upper bound is x, there is no common lower bound.



trees



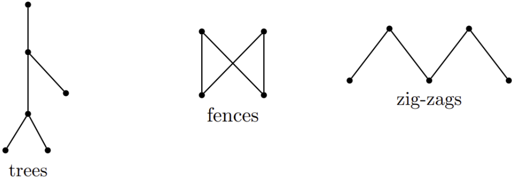
fences



zig-zags

Realizations of posets

The **realisation** of a poset of dimension 1 is the disjoint union of $\mathcal{Q}$ and $\amalg_{x \in \mathcal{Q}} \amalg_{y \in \mathcal{P}(x)} (-1, 0)$. It is again of dimension 1.



If $\mathcal{Q}$ is the realization of a poset of dimension 1, then

$$\mathrm{Tame}(\mathcal{Q}, \mathrm{ch}(\mathrm{vect})) \subset \mathrm{Fun}(\mathcal{Q}, \mathrm{ch}(\mathrm{vect}))$$

Abelian	✓	✓
Model	✓	✗

Cofibrant replacements

The model structure on $\mathrm{Tame}(\mathcal{Q}, \mathbf{ch}(\mathbf{vec}))$ is such that each object X admits a unique up to isomorphism **minimal cofibrant replacement** Y i.e. there is $\varphi\colon Y \to X$ such that:

- φ is an epimorphism at each positive degree and a quasi-isomorphism
- Y is free in each degree
- all summands of Y are acyclic

We think about a minimal cofibrant replacement Y as a simplifying approximation of X.

Structure theorem for cofibrant objects

Structure Theorem

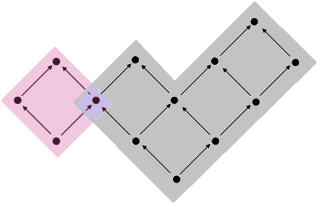
Each cofibrant object in $\mathrm{Tame}\,(\mathcal{Q}, \mathbf{ch}\,(\mathbf{vec}))$ is isomorphic to a direct sum of indecomposables X, each one degreewise-free and nonzero in at most two consecutive degrees with differentials being monomorphisms:

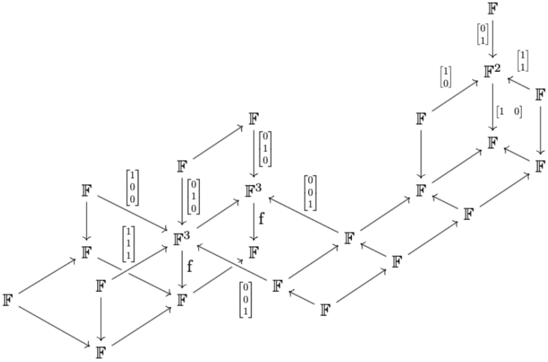
- if X is acyclic, then it is isomorphic to $\mathbb{F}(x,-) \xrightarrow{\mathrm{id}} \mathbb{F}(x,-)$ in degrees i and $i-1$,
- otherwise, there is a unique i such that if $H_i(X) \neq 0$, and X is isomorphic to $P = P_1 \hookrightarrow P_0$, where P is the minimal free resolution of $H_i(X)$ in $\mathrm{Tame}\,(\mathcal{Q}, \mathbf{vec})$.

[Chachólski-Giunti-L.-Tombari: Abelian and model structures on tame functors]

Counter-example for higher dimensional posets

Not true for posets with dimension strictly greater than 1: the following parametrized chain complex is degree-wise free and indecomposable but nonzero in four consecutive degrees.





Conclusions

Take home message:

Parametrized chain complexes are a wealthy source of invariants for persistence theory

Thank you for your attention!

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