Monodromy of supersolvable toric arrangements

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Abelian arrangements

$\mathbb{G}$ a connected abelian Lie group $\quad \Lambda \cong \mathbb{Z}^d \quad T = \mathrm{Hom}(\Lambda, \mathbb{G}) \cong \mathbb{G}^d$

Abelian arrangement: for $\chi_1, \dots, \chi_m \in \Lambda \quad \mathcal{A} = \{H_1, \dots, H_n\}$

where the H_j are the connected components of $\left\{t \in T \,\middle|\, \chi_i \in \ker(t)\right\}$

($H_1, \dots, H_n$ all distinct)

Topology: *complement* $\quad M(\mathcal{A}) = T \setminus \bigcup_{j=1}^n H_j$

Combinatorics: *poset of layers* $\quad \mathcal{P} = \mathcal{P}(\mathcal{A})$

set of all nonempty connected components of $\bigcap_{i \in S} H_i$ for $S \subseteq \{1, \dots, n\}$
order – reverse inclusion rank – codimension

atoms$(\mathcal{P})$: $\quad H_1, \dots, H_n \in \mathcal{A}$

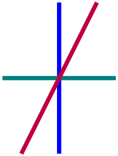
often assume: max'l elts of $\mathcal{P}(\mathcal{A})$ are points in $T = \mathbb{G}^d \quad \operatorname{rank} \mathcal{A} = d$

$\mathbb{G} = \mathbb{C}$ *hyperplane arrangement* $\mathcal{A}$ in $\mathrm{Hom}(\Lambda, \mathbb{C}) \cong \mathbb{C}^d$

$\mathbb{G} = \mathbb{C}^*$ *toric arrangement* $\mathcal{A}$ in $\mathrm{Hom}(\Lambda, \mathbb{C}^*) \cong (\mathbb{C}^*)^d$

Example $\quad \Lambda = \mathbb{Z}^2 \qquad \begin{pmatrix} \chi_1 & \chi_2 & \chi_3 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

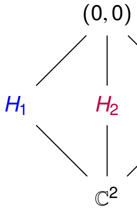
- determines a hyperplane arrangement $\quad \mathscr{A} = \{H_1, H_2, H_3\}$ in $T = \mathbb{C}^2$



$$H_1 = \{2x = 0\}$$

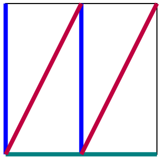
$$H_2 = \{-2x + y = 0\}$$

$$H_3 = \{y = 0\}$$



$\mathcal{P}(\mathscr{A})$

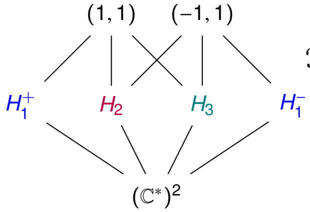
- determines a toric arrangement $\quad \mathscr{A} = \{H_1^{\pm}, H_2, H_3\}$ in $T = (\mathbb{C}^*)^2$



$$H_1^{+} \cup H_1^{-} = \{x^2 = 1\}$$

$$H_2 = \{x^{-2}y = 1\}$$

$$H_3 = \{y = 1\}$$



$\mathcal{P}(\mathscr{A})$

Graphic arrangements / Configuration spaces

Γ graph vertex set $[n] = \{1, 2, \ldots, n\}$ no loops or multiple edges

$\Lambda = \langle v_i \rangle_{i=1}^n \cong \mathbb{Z}^n$ $\mathcal{A}_\Gamma = \left\{ H_{ij} \text{ from } v_i - v_j \in \Lambda \;\middle|\; (i,j) \text{ an edge of } \Gamma \right\}$ in $\mathbb{G}^n$

$M(\mathcal{A}_\Gamma) = \left\{ (x_1, \ldots, x_n) \in \mathbb{G}^n \;\middle|\; x_i \neq x_j \text{ if } (i,j) \text{ an edge of } \Gamma \right\}$

a "partial configuration space"

Γ complete $\implies M(\mathcal{A}_\Gamma) = \mathrm{Conf}_n(\mathbb{G}) = \left\{ (x_1, \ldots, x_n) \in \mathbb{G}^n \;\middle|\; x_i \neq x_j \text{ if } i \neq j \right\}$

ordered configuration space

Fadell-Neuwirth (1962) $\pi\colon \mathrm{Conf}_{n+1}(\mathbb{G}) \to \mathrm{Conf}_n(\mathbb{G})$ is a fiber bundle

fiber $\mathbb{G} \smallsetminus \{n \text{ points}\}$ $(x_1, \ldots, x_n, x_{n+1}) \mapsto (x_1, \ldots, x_n)$

has a section for $\mathbb{G}$ noncompact

equivariant with respect to $\Sigma_n \times 1 \subset \Sigma_{n+1}$ and Σ_n actions

Proposition $\exists$ induced bundle $\pi\colon \mathrm{Conf}^{(n,1)}(\mathbb{G}) \to \mathrm{UConf}_n(\mathbb{G})$

where $\mathrm{Conf}^{(n,1)}(\mathbb{G}) = \mathrm{Conf}_{n+1}(\mathbb{G})/\Sigma_n \times 1$ $\mathrm{UConf}_n(\mathbb{G}) = \mathrm{Conf}_n(\mathbb{G})/\Sigma_n$

has a section for $\mathbb{G}$ noncompact unordered config space

Supersolvability preliminaries

$\mathcal{Q} \subset \mathcal{P} = \mathcal{P}(\mathcal{A})$ corank one subposet

downward-closed and closed under intersection

$\mathcal{Q}$ is an *M-ideal* if for all distinct $H_1, H_2 \in \mathsf{atoms}(\mathcal{P}) \smallsetminus \mathcal{Q}$

each component of $H_1 \cap H_2$ is contained in some $H_3 \in \mathsf{atoms}(\mathcal{Q})$

$\mathcal{Q}$ is a *TM-ideal* if additionally $\forall\ H \in \mathsf{atoms}(\mathcal{P}) \smallsetminus \mathcal{Q}$ and $Y \in \mathcal{Q}$

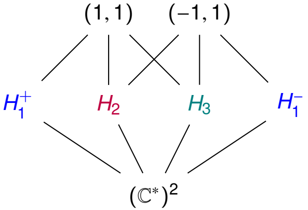
$H \cap Y$ is connected

Example $H_1^{\pm} : \{x^2 = 1\}$, $H_2 : \{y = x^2\}$, $H_3 : \{y = 1\}$ in $T = (\mathbb{C}^*)^2$

$\mathcal{Q} = \{T, H_3\}$ M-ideal

not TM-ideal: $H_2 \cap H_3$ not connected

$\mathcal{Q} = \{T, H_1^+, H_1^-\}$ TM-ideal



Supersolvable abelian arrangements

$\mathcal{A}$ is *(strictly) supersolvable* if there is a maximal chain

$$T = \mathcal{Q}_0 \subsetneq \mathcal{Q}_1 \subsetneq \mathcal{Q}_2 \subsetneq \cdots \subsetneq \mathcal{Q}_{d-1} \subsetneq \mathcal{Q}_{d=\operatorname{rank}\mathcal{P}} = \mathcal{P}$$

with each $\mathcal{Q}_j$ a (T)M-ideal of its successor

Bibby-Delucchi (2022) — Terao $\mathbb{G} = \mathbb{C}$ (1986)

For $\mathcal{A} = \mathcal{A}_d$ supersolvable, there are arrangements $\mathcal{A}_j$ with $\mathcal{Q}_j = \mathcal{P}(\mathcal{A}_j)$ and maps $M(\mathcal{A}_j) \to M(\mathcal{A}_{j-1})$ so that the complement $M(\mathcal{A})$ sits atop a tower of fiber bundles

$$M(\mathcal{A}) \longrightarrow M(\mathcal{A}_{d-1}) \longrightarrow \cdots \longrightarrow M(\mathcal{A}_2) \longrightarrow M(\mathcal{A}_1)$$

The fiber of $M(\mathcal{A}_j) \to M(\mathcal{A}_{j-1})$ is homeomorphic to $\mathbb{G} \smallsetminus \{\ell_j \text{ points}\}$

Example $H_1^{\pm} : \{x^2 = 1\}$, $H_2 : \{y = x^2\}$, $H_3 : \{y = 1\}$ in $T = (\mathbb{C}^*)^2$

$\mathcal{Q}_1 = \{T, H_3\}$ M-ideal

$M(\mathcal{A}) \xrightarrow{\textit{bundle}} M(\mathcal{A}_1) = \mathbb{C} \smallsetminus \{0, 1\}$

fiber $\mathbb{C}^* \smallsetminus \{4 \text{ points}\}$

$\mathcal{Q}_1 = \{T, H_1^+, H_1^-\}$ TM-ideal

$M(\mathcal{A}) \xrightarrow{\textit{bundle}} M(\mathcal{A}_1) = \mathbb{C} \smallsetminus \{0, \pm 1\}$

fiber $\mathbb{C}^* \smallsetminus \{2 \text{ points}\}$

Supersolvable toric arrangement bundles as pullbacks

$\mathcal{A} \subset (\mathbb{C}^*)^d$ supersolvable toric arrangement "over" $\mathcal{B}$ with $\mathcal{P}(\mathcal{B}) = \mathcal{Q}_{d-1}$

$\mathcal{A} \setminus \mathcal{B} = \{H_1, \ldots, H_k\}$ $\quad H_j = \left\{(\mathbf{x}, y) \in (\mathbb{C}^*)^{d-1} \times \mathbb{C}^* \;\middle|\; y^{m_{j,0}} - \underbrace{\mu_j}_{\text{a root of unity}} \mathbf{x}^{\mathbf{m}_j} = 0\right\}$

$f\colon M(\mathcal{B}) \times \mathbb{C} \to \mathbb{C} \quad f(\mathbf{x}, y) = y \prod_{j=1}^{k} (y^{m_{j,0}} - \mu_j \mathbf{x}^{\mathbf{m}_j}) = y^n + \sum_{i=1}^{n} a_i(\mathbf{x}) y^{n-i}$

Features
- continuous coefficient maps $a_i\colon M(\mathcal{B}) \to \mathbb{C}$
- $\forall\, \mathbf{x} \in M(\mathcal{B})$, $f(\mathbf{x}, y) \in \mathbb{C}[y]$ has $n = \ell + 1$ distinct roots
- $\mathcal{A}$ *strictly* supersolvable over $\mathcal{B}$ $\implies$ $f(\mathbf{x}, y) = y \prod_{j=1}^{\ell=k} (y^1 - \mu_j \mathbf{x}^{\mathbf{m}_j})$

Theorem $\mathcal{A}$ supersolvable over $\mathcal{B}$ as above
- bundle $M(\mathcal{A}) \to M(\mathcal{B})$, fiber $\mathbb{C}^* \setminus \{\ell \text{ pts}\} = \mathbb{C} \setminus \{n \text{ pts}\}$, is the pullback of $\pi\colon \operatorname{Conf}^{(n,1)}(\mathbb{C}) \to \operatorname{UConf}_n(\mathbb{C})$ along $\mathbf{a}\colon M(\mathcal{B}) \to \operatorname{UConf}_n(\mathbb{C})$
  $\mathbf{x} \mapsto \{\text{roots of } f(\mathbf{x}, y) \in \mathbb{C}[y]\}$
- $\mathcal{A}$ strictly supersolvable over $\mathcal{B}$ $\implies$ $M(\mathcal{A}) \to M(\mathcal{B})$ is the pullback of $\pi\colon \operatorname{Conf}_{n+1}(\mathbb{C}) \to \operatorname{Conf}_n(\mathbb{C})$ along $\mathbf{b}\colon M(\mathcal{B}) \to \operatorname{Conf}_n(\mathbb{C})$
  $\mathbf{x} \mapsto (0, \mu_1 \mathbf{x}^{\mathbf{m}_1}, \ldots, \mu_k \mathbf{x}^{\mathbf{m}_\ell})$

Example $H_1^{\pm}: \{x^2 = 1\}$, $H_2: \{y = x^2\}$, $H_3: \{y = 1\}$ in $(\mathbb{C}^*)^2$

$\mathcal{Q}_1 = \{T, H_3\}$ M-ideal

$$\begin{tikzcd} M(\mathcal{A}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Conf}^{(5,1)}(\mathbb{C}) \arrow[d] \\ M(\mathcal{B}) \arrow[r, "\mathbf{a}"] & \mathrm{UConf}_5(\mathbb{C}) \end{tikzcd}$$

$M(\mathcal{B}) = \mathbb{C} \smallsetminus \{0, 1\}$

$\mathbf{a}\colon\ y \mapsto \{0, \pm\sqrt{y}, \pm 1\}$

$\mathcal{Q}_1 = \{T, H_1^+, H_1^-\}$ TM-ideal

$$\begin{tikzcd} M(\mathcal{A}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Conf}_4(\mathbb{C}) \arrow[d] \\ M(\mathcal{B}) \arrow[r, "\mathbf{b}"] & \mathrm{Conf}_3(\mathbb{C}) \end{tikzcd}$$

$M(\mathcal{B}) = \mathbb{C} \smallsetminus \{0, \pm 1\}$

$\mathbf{b}\colon\ x \mapsto (0, x^2, 1)$

General supersolvable / strictly supersolvable toric arrangements

$\mathcal{A} = \mathcal{A}_d \subset (\mathbb{C}^*)^d$ supersolvable / strictly supersolvable

$$\begin{tikzcd} M(\mathcal{A}_j) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Conf}^{(n_j,1)}(\mathbb{C}) \arrow[d] \\ M(\mathcal{A}_{j-1}) \arrow[r, "\mathbf{a}_j"] & \mathrm{UConf}_{n_j}(\mathbb{C}) \end{tikzcd}$$

$$\begin{tikzcd} M(\mathcal{A}_j) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Conf}_{n_j+1}(\mathbb{C}) \arrow[d] \\ M(\mathcal{A}_{j-1}) \arrow[r, "\mathbf{b}_j"] & \mathrm{Conf}_{n_j}(\mathbb{C}) \end{tikzcd}$$

$\rightsquigarrow$ sequences $\vec{\mathbf{a}} = (\mathbf{a}_2, \dots, \mathbf{a}_d)$ / $\vec{\mathbf{b}} = (\mathbf{b}_2, \dots, \mathbf{b}_d)$

Fundamental group

Theorem For $\mathcal{A} = \mathcal{A}_d$ a supersolvable toric arrangement $\quad n_1 = 1 + |\mathcal{A}_1|$

$M(\mathcal{A})$ is a $K(G,1)$ $\qquad G = \pi_1(M(\mathcal{A})) = F_{n_d} \rtimes \big(F_{n_{d-1}} (\rtimes \cdots \rtimes (F_{n_2} \rtimes F_{n_1}\big)$

semidirect product structure is determined by $\vec{\mathbf{a}}_\sharp = \big((\mathbf{a}_2)_\sharp, \ldots, (\mathbf{a}_d)_\sharp\big)$

$(\mathbf{a}_j)_\sharp \colon \pi_1(M(\mathcal{A}_{j-1})) \longrightarrow \pi_1(\mathrm{UConf}_{n_j}(\mathbb{C})) = B_{n_j}$ braid group

and the classical Artin representation $B_n \longrightarrow \mathrm{Aut}(F_n)$

for $\mathcal{A}$ supersolvable over $\mathcal{B}$ as before

$$
\begin{tikzcd}
1 \arrow[r] & F_n \arrow[r] \arrow[d, "="] & \pi_1(M(\mathcal{A})) \arrow[r] \arrow[d] & \pi_1(M(\mathcal{B})) \arrow[r] \arrow[d, "\mathbf{a}_\sharp"] & 1 \\
1 \arrow[r] & F_n \arrow[r] & \pi_1(\mathrm{Conf}^{(n,1)}(\mathbb{C})) \arrow[r] & \pi_1(\mathrm{UConf}_n(\mathbb{C})) \arrow[r] \arrow[l, bend right] & 1
\end{tikzcd}
.
$$

$\pi_1(M(\mathcal{B}))$ acts on F_n via $\pi_1(M(\mathcal{B})) \xrightarrow{\mathbf{a}_\sharp} B_n \xrightarrow{Artin} \mathrm{Aut}(F_n)$

Corollary $G = \pi_1(M(\mathcal{A}))$ is a linear group

Braid group / Artin representation

$$\underset{\text{braid group}}{B_n = \pi_1(\mathrm{UConf}_n(\mathbb{C}))} = \Big\langle \underset{1\le i<n}{\sigma_i} \;\Big|\; \underset{|i-j|\ge 2}{\sigma_i\sigma_j = \sigma_j\sigma_i}, \; \underset{1\le i<n-1}{\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}} \Big\rangle$$

$$B_n \xrightarrow{Artin} \mathrm{Aut}(F_n) \qquad \text{if } F_n = \Big\langle \underset{1\le j\le n}{z_j} \Big\rangle \qquad \sigma_i(z_j) = \begin{cases} z_i z_{i+1} z_i^{-1} & j = i \\ z_i & j = i+1 \\ z_j & \text{else} \end{cases}$$

$$\underset{\text{pure braid group}}{P_n = \pi_1(\mathrm{Conf}_n(\mathbb{C}))} = \Big\langle \underset{1\le i<j\le n}{a_{ij}} = \sigma_{j-1}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-1}^{-1} \;\Big|\; \textit{relations} \Big\rangle$$

$$P_n \hookrightarrow B_n \xrightarrow{Artin} \mathrm{Aut}(F_n) \qquad a_{ij}(z_k) = \begin{cases} z_i z_j z_k z_j^{-1} z_i^{-1} & k \in \{i,j\} \\ z_i z_j z_i^{-1} z_j^{-1} z_k z_j z_i z_j^{-1} z_i^{-1} & i < k < j \\ z_k & \text{else} \end{cases}$$

Theorem $\mathcal{A}$ strictly supersolvable $\implies$ G an *almost-direct product*

$\Big($action of $\rtimes_{i=1}^{j} F_{n_i}$ on $H_1(F_{n_k})$ is trivial for each j, k with $1 \le j < k \le r\Big)$

determined by Artin & $\overrightarrow{\mathbf{b}}_\sharp$ $\qquad (\mathbf{b}_j)_\sharp \colon \pi_1(M(\mathcal{A}_{j-1})) \to \pi_1(\mathrm{Conf}_{n_j}(\mathbb{C})) = P_{n_j}$

Homology

$\mathbb{Z}$ coefficients for H_* and H^*

Corollary

B-D (2022)

$\mathcal{A}$ strictly supersolvable $\implies$ $H_*(M(\mathcal{A}))$ torsion free

Betti numbers given by $\sum_{i=0}^{d} \operatorname{rank} H_i(M(\mathcal{A}))t^i = \prod_{j=1}^{d}(1+n_j t)$

Example

$H_1^{\pm}:\{x^2=1\}$, $H_2:\{y=x^2\}$, $H_3:\{y=1\}$ in $T=(\mathbb{C}^*)^2$

$\mathcal{Q}_1 = \{T, H_3\}$ M-ideal

$\mathcal{B} = \{H_3\}$ $\quad M(\mathcal{B}) = \mathbb{C} \setminus \{0,1\}$

$\mathbf{a}: M(\mathcal{B}) \longrightarrow \mathrm{UConf}_5(\mathbb{C})$

$\quad y \mapsto \{0, \pm\sqrt{y}, \pm 1\}$

$\mathbf{a}_\sharp: F_2 \longrightarrow B_5 \quad \mathrm{y}_0 \mapsto \sigma_2\sigma_3\sigma_2$

$\quad \mathrm{y}_1 \mapsto \sigma_1^2\sigma_4^2$

$\pi_1(M(\mathcal{A})) = F_5 \rtimes F_2$ semidirect

not almost-direct

$\mathcal{Q}_1 = \{T, H_1^+, H_1^-\}$ TM-ideal

$\mathcal{B} = \{H_1\}$ $\quad M(\mathcal{B}) = \mathbb{C} \setminus \{0,\pm 1\}$

$\mathbf{b}: M(\mathcal{B}) \longrightarrow \mathrm{Conf}_3(\mathbb{C})$

$\quad x \mapsto (0, x^2, 1)$

$\mathbf{b}_\sharp: F_3 \longrightarrow P_3 \quad \mathrm{x}_0 \mapsto a_{12}^2 \quad \mathrm{x}_1 \mapsto a_{23}$

$\quad \mathrm{x}_{-1} \mapsto a_{12}a_{23}a_{12}^{-1}$

$\pi_1(M(\mathcal{A})) = F_3 \rtimes F_3$ almost-direct

so $H_1(M(\mathcal{A})) = \mathbb{Z}^6$ $\;H_2(M(\mathcal{A})) = \mathbb{Z}^9$

LCS=Lower Central Series Lie algebra

G discrete group LCS: $G_1 = G$ $G_{k+1} = [G_k, G]$ $\mathfrak{g}_k = G_k/G_{k+1}$
subgroups quotients

Lie algebra $\mathfrak{g} = \mathfrak{g}(G) = \bigoplus_{k\geq 1} \mathfrak{g}_k$ bracket from $[u, v] = uvu^{-1}v^{-1}$ in G

Example $G = F_n$ free group $\Longrightarrow$ $\mathfrak{g} = \mathbb{L}[n]$ free Lie algebra

Theorem For $\mathcal{A} = \mathcal{A}_d$ strictly supersolvable toric & $G = \pi_1(M(\mathcal{A}))$
$\mathfrak{g}_{\mathcal{A}} = \mathfrak{g}(G) = \mathbb{L}[n_d] \rtimes \big(\mathbb{L}[n_{d-1}](\rtimes \cdots \rtimes (\mathbb{L}[n_2] \rtimes \mathbb{L}[n_1]))\big)$ semidirect product

structure from $\vec{\beta} = (\beta_2, \ldots, \beta_d)$ $H_1(M(\mathcal{A}_{j-1})) \xrightarrow{\beta_j = (\mathbf{b}_j)_*} H_1(\mathrm{Conf}_{n_j}(\mathbb{C}))$

and the "infinitesimal pure braid relations" bracket rels in $\mathfrak{g}(P_n)$

Corollary Kohno P_n, Falk-Randell $\mathbb{G} = \mathbb{C}$ (1985), B-D $\mathbb{G} = \mathbb{C}^*$ (2022)

$$\prod_{k=1}^{\infty}(1 - t^k)^{\varphi_k} = \prod_{j=1}^{d}(1 - n_j t) = \sum_{i=0}^{d} \operatorname{rank} H_i(G; \mathbb{Z})(-t)^i \quad \text{in } \mathbb{Z}[[t]]$$

where φ_k is the rank of the k-th lower central series quotient of G

LCS Lie algebra illustrations

$U = [u]$ homology class

Kohno (1985) $G = P_n = \pi_1(\mathrm{Conf}_n(\mathbb{C}))$ pure braid group $\quad \mathfrak{g}_n = \mathfrak{g}(P_n)$

gens: $\underset{1\le i<j\le n}{A_{ij} = [a_{ij}]}$ rels: $\underset{i,j,k,l \text{ distinct}}{[A_{ij}, A_{kl}] = 0}$ $\quad \underset{i<j<k,\ q=i,j}{[A_{qk}, A_{ij} + A_{ik} + A_{jk}] = 0}$

$\Longrightarrow \quad \mathfrak{g}_{n+1} = \mathbb{L}[n] \rtimes \mathfrak{g}_n \quad \mathfrak{g}_n$ action on $\mathbb{L}[n] = \langle \underset{1\le i\le n}{Z_i} \rangle$ given by

$$[A_{ij}, Z_q] = [Z_q, (\delta_{iq} + \delta_{jq})(Z_i + Z_j)]$$

Example $\quad H_1^{\pm} : \{x^2 = 1\}$, $H_2 : \{y = x^2\}$, $H_3 : \{y = 1\}$ in $T = (\mathbb{C}^*)^2$

$\mathfrak{Q}_1 = \{T, H_1^+, H_1^-\}$ TM-ideal $\quad \mathcal{B} = \{H_1^+, H_1^-\} \quad M(\mathcal{B}) = \mathbb{C} \smallsetminus \{0, \pm 1\}$

$M(\mathcal{B}) \xrightarrow{\mathbf{b}} \mathrm{Conf}_3(\mathbb{C}) \qquad F_3 \xrightarrow{\mathbf{b}_\sharp} P_3 \qquad H_1(M(\mathcal{B})) \xrightarrow{\beta = \mathbf{b}_*} H_1(\mathrm{Conf}_3(\mathbb{C}))$

$x \mapsto (0, x^2, 1)$

$$\begin{array}{lll} \mathrm{x}_0 \mapsto a_{12}^2 & \qquad & X_0 \mapsto 2A_{12} \\ \mathrm{x}_1 \mapsto a_{23} & & X_1 \mapsto A_{23} \\ \mathrm{x}_{-1} \mapsto a_{12}a_{23}a_{12}^{-1} & & X_{-1} \mapsto A_{23} \end{array}$$

$\mathfrak{g}_{\mathcal{A}} = \mathfrak{g}(\pi_1(M(\mathcal{A}))) = \mathbb{L}[3] \rtimes \mathfrak{g}_{\mathcal{B}} \quad \mathfrak{g}_{\mathcal{B}} = \langle X_0, X_1, X_{-1} \rangle$ acts on $\mathbb{L}[3] = \langle Z_i \rangle$:

$[X, Z] = [\beta(X), Z] \quad$ e.g. $[X_0, Z_i] = [2A_{12}, Z_i] = 2[Z_i, (\delta_{i1} + \delta_{i2})(Z_1 + Z_2)]$

Cohomology

Theorem For $\mathcal{A} = \mathcal{A}_d$ a strictly supersolvable toric arrangement

$H^*(M(\mathcal{A})) \cong \bigwedge H^1(M(\mathcal{A}))\big/\mathcal{J}$ $\quad$ $\mathcal{J}$ determined by $\vec{\beta} = (\beta_2, \dots, \beta_d)$

rationally Koszul $\quad$ $\beta_j := (\mathbf{b}_j)_* \colon H_1(M(\mathcal{A}_{j-1})) \longrightarrow H_1(\mathrm{Conf}_{n_j}(\mathbb{C}))$

$\pi_1(M(\mathcal{A})) \xrightarrow[\text{abelianize}]{\mathfrak{a}} H_1(M(\mathcal{A}))$ induces $H_2(M(\mathcal{A})) \xrightarrow{\mathfrak{a}_*} \bigwedge^2 H_1(M(\mathcal{A}))$

$$\mathcal{J} = \left\langle \ker \left[\mathrm{Hom}(\textstyle\bigwedge^2 H_1(M(\mathcal{A})), \mathbb{Z}) \xrightarrow{\text{dual to } \mathfrak{a}_*} \mathrm{Hom}(H_2(M(\mathcal{A})), \mathbb{Z}) \right] \right\rangle$$

LCS Lie algebra relations encode $H_2(M(\mathcal{A})) \xrightarrow{\mathfrak{a}_*} \bigwedge^2 H_1(M(\mathcal{A}))$ & $\mathcal{J}$

Example $H_1^{\pm} : \{x^2 = 1\}$, $H_2 : \{y = x^2\}$, $H_3 : \{y = 1\}$ in $T = (\mathbb{C}^*)^2$

$\mathfrak{g}_{\mathcal{A}}$ gens: $X_0, X_1, X_{-1}, Z_1, Z_2, Z_3$

rels: $[X_0, Z_1] - 2[Z_1, Z_2]$, $\quad [X_0, Z_2] + 2[Z_1, Z_2]$, $\quad [X_0, Z_3]$

$[X_{\pm 1}, Z_1]$, $\quad [X_{\pm 1}, Z_2] - [Z_2, Z_3]$, $\quad [X_{\pm 1}, Z_3] + [Z_2, Z_3]$

$H^*(M(\mathcal{A})) \cong \bigwedge \langle X_0, X_1, X_{-1}, Z_1, Z_2, Z_3 \rangle \big/ \mathcal{J}$

$$\mathcal{J} = \left\langle \begin{matrix} X_0 \wedge X_1, \quad X_0 \wedge X_{-1}, \quad X_1 \wedge X_{-1}, \quad Z_1 \wedge Z_2 + 2X_0 \wedge Z_1 - 2X_0 \wedge Z_2 \\ Z_2 \wedge Z_3 + X_1 \wedge Z_2 - X_1 \wedge Z_3, \quad Z_2 \wedge Z_3 + X_{-1} \wedge Z_2 - X_{-1} \wedge Z_3 \end{matrix} \right\rangle$$

Type C

Type C toric arrangement $\mathcal{C}_n$ in $(\mathbb{C}^*)^n \subset \mathbb{C}^n$ hypersurfaces given by

$$x_i = 0,\ x_i = 1,\ x_i = -1 \Big\} 1 \le i \le n \qquad x_j - x_i = 0,\ x_j - x_i^{-1} = 0 \Big\} 1 \le i < j \le n$$

complement $M(\mathcal{C}_n) = \mathbb{C}^n \smallsetminus \bigcup\{\text{ these hypersurfaces }\}$

$\mathcal{C}_n$ is strictly supersolvable e.g., $\mathbf{b}\colon M(\mathcal{C}_{n-1}) \to \mathrm{Conf}_{2n+1}(\mathbb{C})$ given by

$$\mathbf{b}\colon (x_1, \dots, x_{n-1}) \mapsto (0, 1, -1, x_1, \dots, x_{n-1}, x_1^{-1}, \dots, x_{n-1}^{-1})$$

Cohomology E exterior algebra generated by degree one elements

$$\zeta_i,\ \rho_i,\ \eta_i \Big\} 1 \le i \le n \qquad \text{and} \qquad \alpha_{ij},\ \beta_{ij} \Big\} 1 \le i < j \le n$$

Theorem $H^*(M(\mathcal{C}_n)) \cong E/\mathcal{J}$ $\mathcal{J}$ generated by

$$\zeta_i\rho_i, \quad \zeta_i\eta_i, \quad \rho_i\eta_i \Big\} 1 \le i \le n$$

$$\left.\begin{array}{lll} (\zeta_j - \zeta_i)(\alpha_{ij} - \zeta_i), & (\rho_j - \rho_i)(\alpha_{ij} - \rho_i), & (\eta_j - \eta_i)(\alpha_{ij} - \eta_i), \\ (\rho_j - \rho_i + \zeta_i)(\beta_{ij} - \rho_j), & (\eta_j - \eta_i + \zeta_i)(\beta_{ij} - \eta_j), & \\ (\zeta_j + \zeta_i)(\beta_{ij} + \zeta_i), & (\alpha_{ij} + \zeta_i - \rho_i - \eta_i)(\beta_{ij} - \alpha_{ij}) & \end{array}\right\} 1 \le i < j \le n$$

$$\left.\begin{array}{ll} (\alpha_{ik} - \alpha_{ij})(\alpha_{jk} - \alpha_{ij}), & (\alpha_{ik} - \zeta_i + \zeta_j - \beta_{ij})(\beta_{jk} - \beta_{ij}), \\ (\beta_{ik} - \beta_{ij})(\alpha_{jk} - \beta_{ij}), & (\beta_{ik} + \zeta_i + \zeta_j - \alpha_{ij})(\beta_{jk} - \beta_{ik}) \end{array}\right\} 1 \le i < j < k \le n$$

Type C Resonance

First resonance variety of $A = \bigoplus_{k\geq 0} A^k = H^*(M(\mathcal{C}_n);\mathbb{C})$

$$\mathcal{R}^1(A) = \left\{ a \in A^1 \,\middle|\, \exists\, b \in A^1 \text{ with } b \neq \lambda a \text{ and } ab = 0 \right\} \qquad \lambda \in \mathbb{C}$$

Yongho Lee   work in progress

The first resonance variety of $A = H^*(M(\mathcal{C}_n);\mathbb{C})$ contains (only?)

$$\underbrace{\binom{n}{1} + \binom{n}{2}}_{3\text{-dimensional}} + \underbrace{16 \cdot \binom{n}{2} + 25 \cdot \binom{n}{3} + 9 \cdot \binom{n}{4}}_{2\text{-dimensional}} \quad \text{linear components}$$

Sample components

boring 3-dim'l component: $\langle\, \zeta_i,\ \rho_i,\ \eta_i \,\rangle$

exciting 3-dim'l component: $\langle\, \rho_i + \eta_j - \beta_{ij},\ \eta_i + \rho_j - \beta_{ij},\ \alpha_{ij} - \beta_{ij} \,\rangle$ C_2

boring 2-dim'l component: $\langle\, \alpha_{ik} - \alpha_{ij},\ \alpha_{jk} - \alpha_{ij} \,\rangle$

exciting 2-dim'l component: $\langle\, \zeta_i + 2\rho_j - \alpha_{ij} - \beta_{ij},\ 2\rho_i + \zeta_j - \alpha_{ij} - \beta_{ij} \,\rangle$ B_2

Consequence $\operatorname{corank} \pi_1(M(\mathcal{C}_n)) = 3$

Topological complexity

$X^I = \{\gamma \colon I \to X \text{ continuous}\}$ free paths on X compact/open topology

$\mathrm{ev} \colon X^I \to X \times X$ $\mathrm{ev}(\gamma) = (\gamma(0), \gamma(1))$ fibration

The *topological complexity* of X (path connected) is the minimal number $\mathrm{TC}(X) = k$ so that $X \times X = U_1 \cup U_2 \cup \cdots \cup U_k$ with each U_i open and $\exists$ continuous section $s_i \colon U_i \to X^I$ $\mathrm{ev} \circ s_i = \mathrm{id}_{U_i}$

Motivation for X the space of all states of a mechanical system $\mathrm{TC}(X)$ measures the complexity of the *motion planning problem* in X

Theorem $\mathcal{A} \subset (\mathbb{C}^*)^d$ a rank r strictly supersolvable toric arrangement $\Longrightarrow$ the complement has topological complexity $\mathrm{TC}(M(\mathcal{A})) = d + r + 1$

uses

- $M(\mathcal{A}) \cong M(\mathcal{A}') \times (\mathbb{C}^*)^{d-r}$ $\mathcal{A}'$ rank r in $(\mathbb{C}^*)^r$ from Callegaro–D'Adderio–Delucchi–Migliorini–Pagaria (2020)
- structure of $\pi_1(M(\mathcal{A}))$ ...