My abductions, deductions and (not only semiotic) mediations with Maria Alessandra

Ferdinando Arzarello

Pisa, March 21, 2025

Maria Alessandra represents one of the most important and significant representatives of the Italian way to mathematics education, as it has developed in the last decades.

For this, all our community is deeply grateful to her.

She was the first to undertake paths that many researchers have subsequently followed, encouraged by her example.

I recall:

- relevant interactions with illustrious foreign scholars in the field of mathematics education, starting from her doctoral thesis advisor, E. Fischbein;
- continuous effective contact with the reality of school, of how mathematics lives (or dies) in it, and the theoretical reflection that derives from these contacts;

→

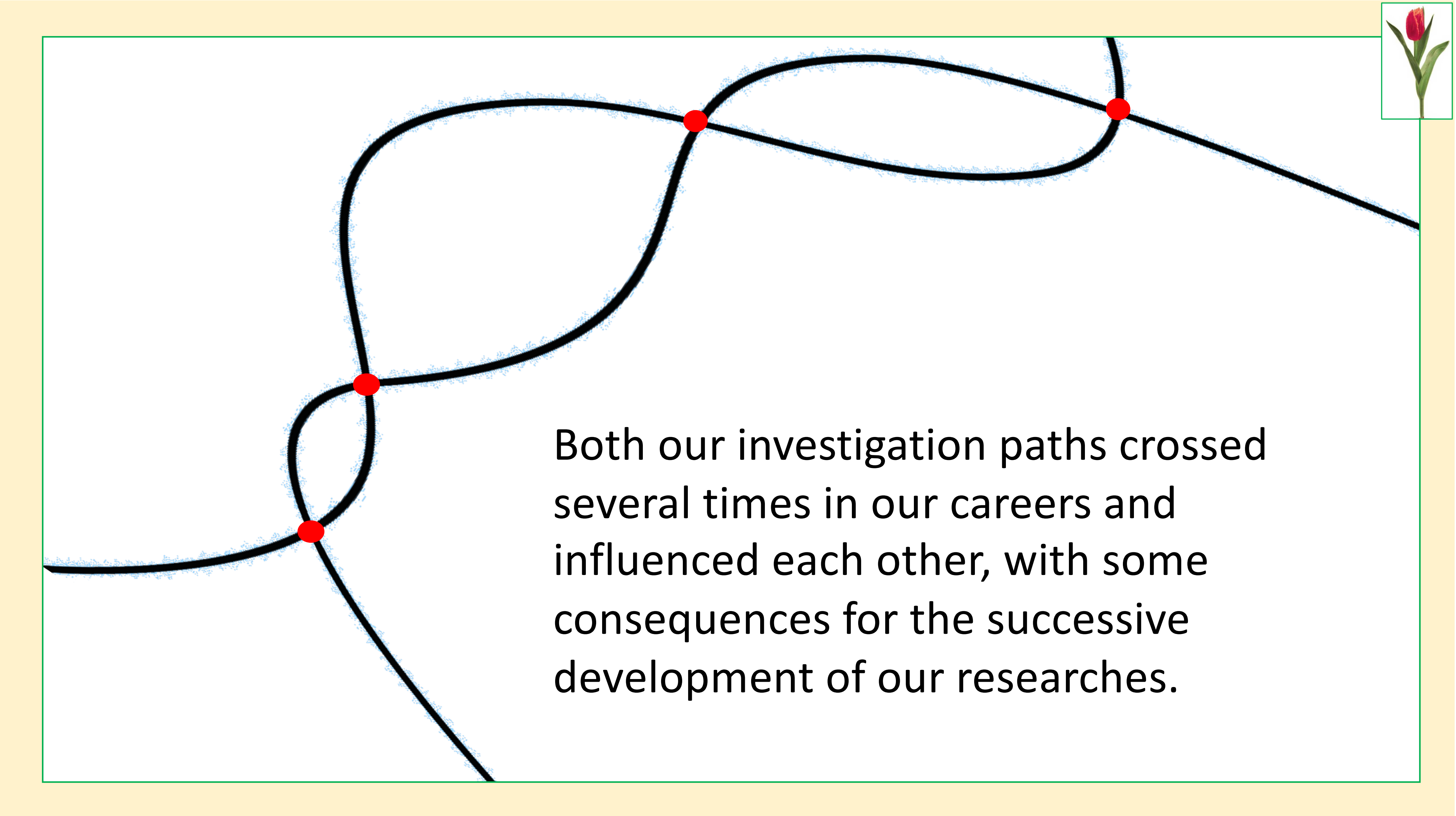
→

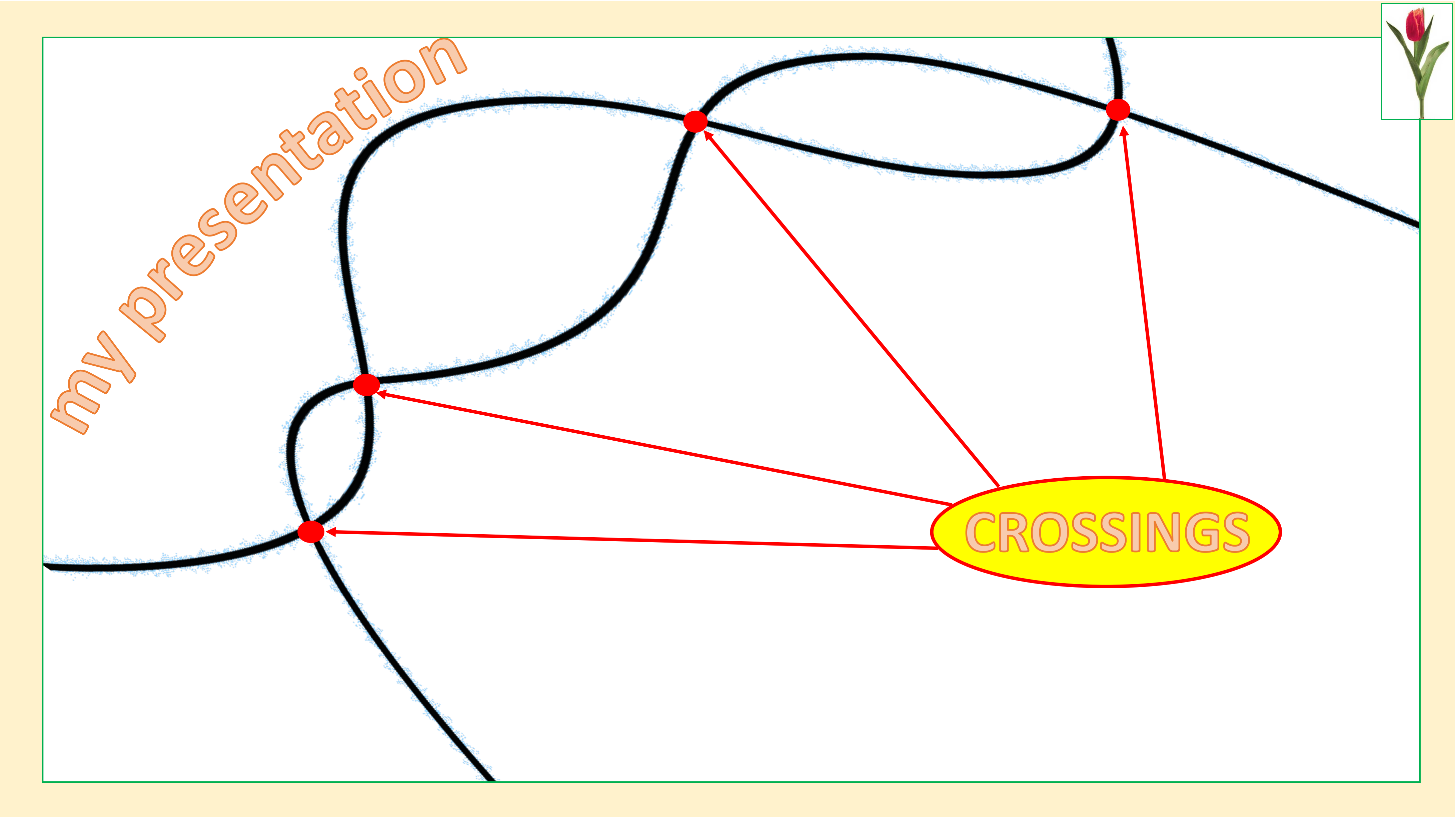
- attention and care for her students: it is no coincidence that many have become brilliant researchers;

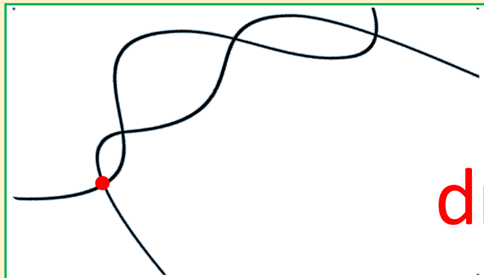
- important collaborations with Italian scholars, including her students, who in the meantime had 'grown’ scientifically;

- attention to the organizational aspects of the community, from when she was secretary of the CIIM in the 1980s to the many important international and national roles she subsequently held.

⋮


Both our investigation paths crossed several times in our careers and influenced each other, with some consequences for the successive development of our researches.


my presentation
CROSSINGS



CROSSING 1

dragging modalities and schemes in DGS (Cabri)

- Arzarello, F., Gallino, G., Micheletti, C., Olivero, F., Paola, D. & Robutti, O.: 1998, Dragging in Cabri and modalities of transition from conjectures to proofs in geometry, Proc. of PME XXII, Stellenbosh, South Africa, v. 2, 32-39.
- Arzarello F.: 2000, 'Inside and Outside: Spaces, Times and Language in Proof Production', in: Proceedings of PME XXIV, Hiroshima, Japan, 1, 23-38.
- Arzarello, F.: 2001, Dragging, perceiving and measuring: physical practices and theoretical exactness in Cabri-environments, Proc. Cabriworld 2, Montreal, Plenary Lecture.
- Baccaglini-Frank, A., Mariotti, M. A., & Antonini, S. (2009). Different perceptions of invariants and generality of proof in dynamic geometry. In Tzekaki, M., & Sakonidis, H. (Eds.), In Proceedings of the 33rd conference of the IGPME, (Vol. 2, pp. 89–96), Thessaloniki, Greece.
- Baccaglini-Frank, A., Mariotti M.A. (2010). Generating Conjectures in Dynamic Geometry: The Maintaining Dragging Model. *International Journal of Computers for Mathematical Learning,* 15(3):225-253.

We both:

- investigated cognitive processes that occur during the phase of conjecture-generation in the solution of open problems: they are associated with specific ‘‘uses’’ of the dragging tool (‘‘dragging schemes’’).

- elaborated models apt to describe and focus on certain possible steps of the cognitive processes that may occur when students engage in explorations within a dynamic geometry system that involve the use of dragging.

As to these issues we shared common methodologies. For example, basing on Labordes’ and others’ research, first I pointed out the following modalities:

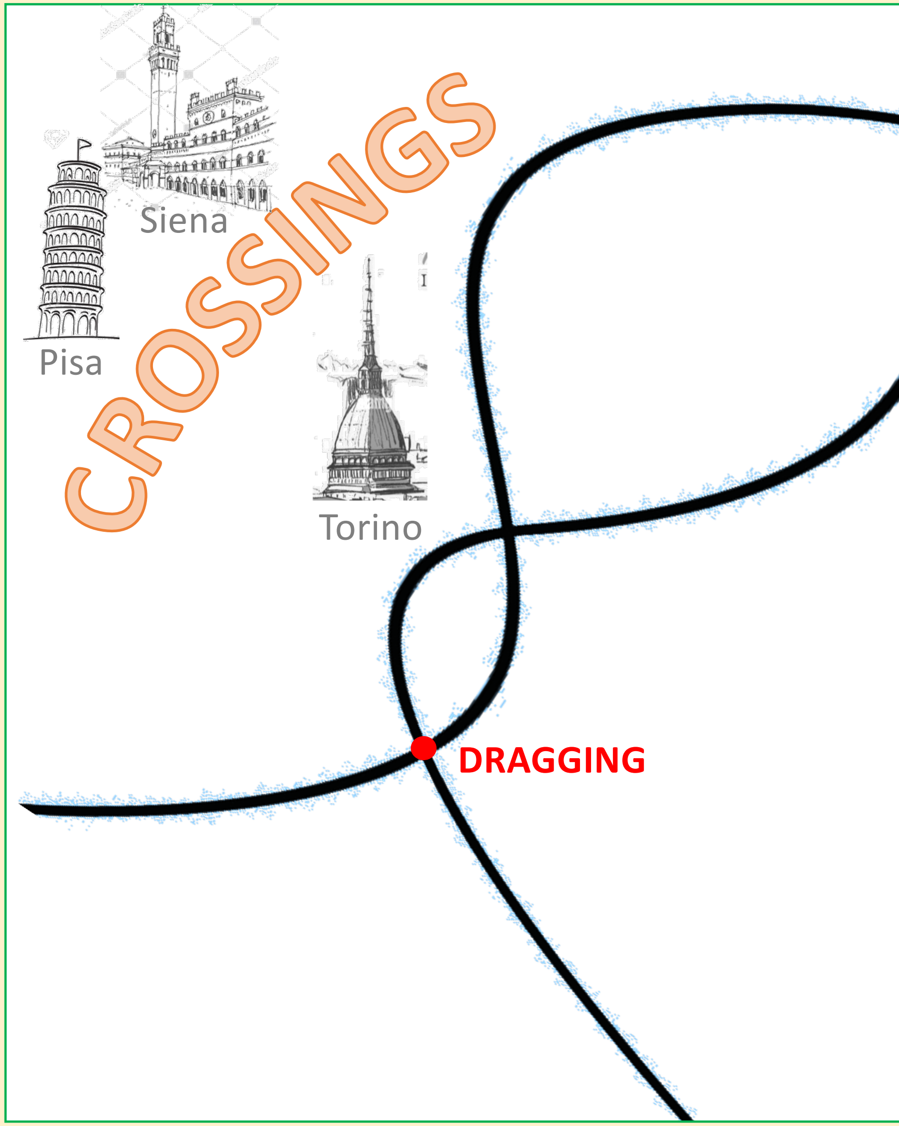
- *Wandering dragging*: moving the basic points on the screen randomly, without a plan, in order to discover interesting configurations or regularities in the drawings.
- *Bound dragging*: moving a semi-dragable point (already linked to an object).
- *Guided dragging*: dragging the basic points of a drawing in order to give it a particular shape.
- *Dummy locus dragging*: moving a basic point so that the drawing keeps a discovered property; the point which is moved follows a path, even if the users do not realise this: the locus is not visible and does not 'speak' to the students, who do not always realise that they are dragging along a locus.

I had used the terminology ‘‘dragging modalities’’, which were not conceived within the instrumentation approach.
On the contrary, other researchers (Leung et al. 2006, 2008; Strässer 2009) analyzed the use of dragging under the lens of the instrumental approach so that dragging became an artefact, which, through instrumental genesis, could support the task of generating a conjecture.
M.A. & A.B.-F. considered dragging as an artefact and placed its users in the context of solving a problem (task): so they could identify specific utilization schemes associated to its use.

Consequently, M.A. & A.B.-F. introduced the important distinction between dragging modalities and dragging utilization schemes, in order to separate what might be observed externally as a particular way of dragging (dragging modality) from the description of an internal mental construct of the solver (dragging utilization scheme) associated to a particular way of dragging.

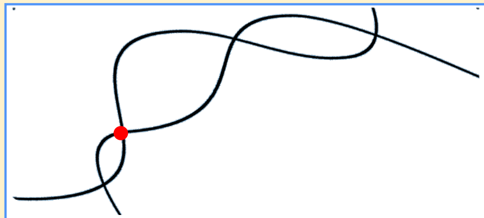
They could so identify the following four typologies of dragging:

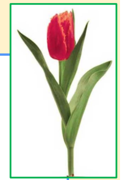
- *wandering/random dragging*: randomly dragging a base point on the screen, looking for interesting configurations or regularities of the Cabri-figure;
- *maintaining dragging*: dragging a base point so that the Cabri-figure maintains a certain property;
- *dragging with trace activated*: dragging a base point with the trace activated;
- *dragging test*: dragging base points to see whether the constructed figure maintains the desired properties. In this mode it can be useful to make a new construction or redefine a point on an object to test a formulated conjecture.



"Besides being able to handle the software, new abilities are demanded from the pupils; above all abilities in relation to 'meaningful experiments', for instance the control of parameters in an experiment and the interpretation of its outcomes. [...] and the challenge for the pupils is to make sense of these outcomes". (p. 185).

Hölzl, R. How does 'dragging' affect the learning of geometry. *Int. Journ. Comput. Math. Learning* **1**, 169–187 (1996)



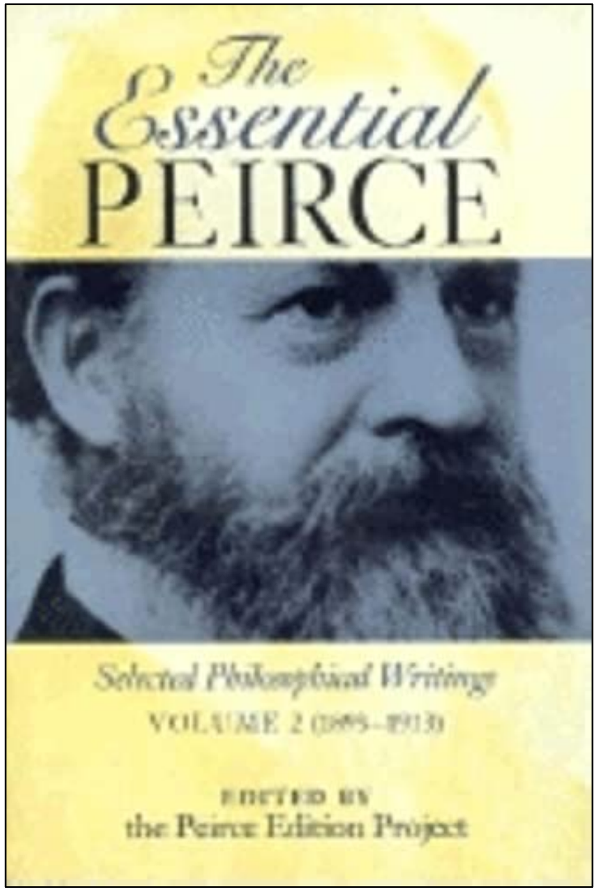


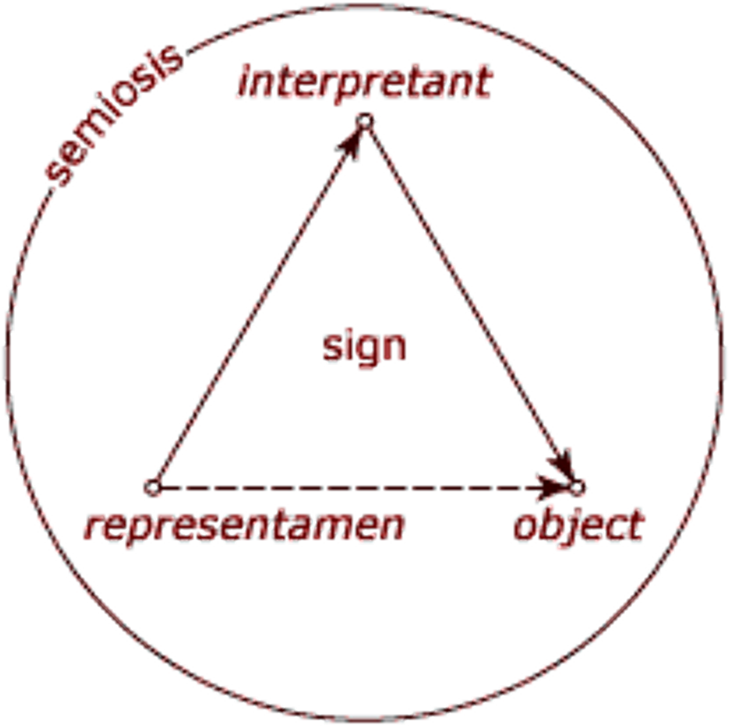
CROSSING 2
from dragging to abductions in DGS (Cabri)

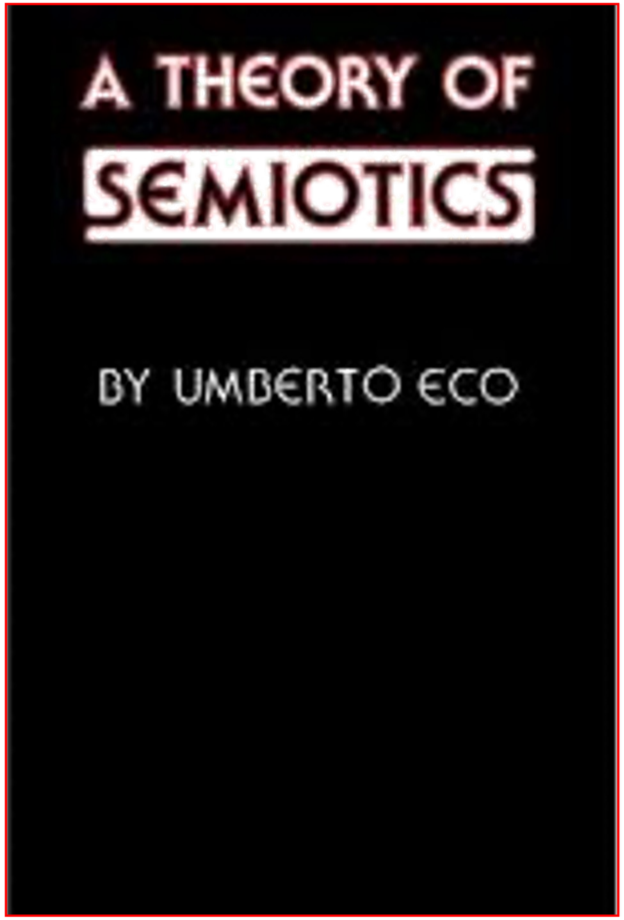
- Baccaglini-Frank, A., Mariotti, M. A., & Antonini, S. (2009). Different perceptions of invariants and generality of proof in dynamic geometry. In Tzekaki, M., & Sakonidis, H. (Eds.), In Proceedings of the 33rd conference of the IGPME, (Vol. 2, pp. 89–96), Thessaloniki, Greece.
- Baccaglini-Frank, A., Mariotti M.A. (2010). Generating Conjectures in Dynamic Geometry: The Maintaining Dragging Model**.** *Intern. Journ. Computers for Math. Learn.,* 15(3):225-253.
- Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practices in cabri environments. *ZDM*, 34(3), 66–72.
- Baccaglini-Frank, A., Mariotti, M. A. (2011). Conjecture-generation through Dragging and Abduction in Dynamic Geometry. In: Méndez-Vilas (Ed.), A.. *Education in a technological world: communicating current and emerging research and technological efforts*. Formatex, Spain, pp. 100-107.
- Arzarello, F., Bartolini Bussi, M.G., Leung, A., Mariotti, M.A., Stevenson, I. (2012). Experimental approaches to theoretical thinking the mathematics classroom: artefacts and proofs. In: Hanna, G., de Villiers, M. (eds) *Proof and Proving in Mathematics Education*. New ICMI Study Series, vol 15. Springer, Dordrecht. pp. 97-143.

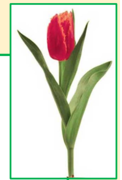
Seminars of the ‘Abduziosi’ in Siena

(after that Maria Alessandra moved from Pisa to that university)









I had pointed out that dragging practises can be framed within a cognitive evolution back and forth from perceptions to abstract ideas, underlying that there are two main processes, which can be differently faded according to the concrete situation (Saada-Robert, 1989; Olivero, 1999; Arzarello, 2000):

- *ascending processes*, from drawings to theory, in order to explore freely a situation, looking for regularities, invariants, etc.
- *descending processes* from theory to drawings, in order to validate or refute conjectures, to check properties, etc.

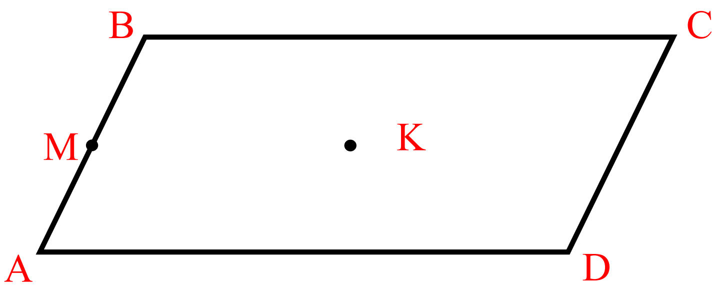
Ascending and descending processes shown by dragging practises in Cabri reveal cognitive shifts from the perceptual level to the theoretical one and back in students' mathematical activity.

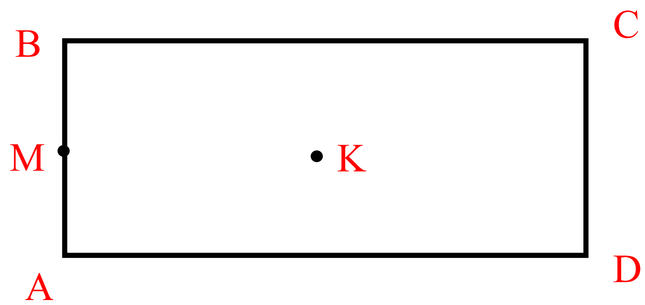
Ascending and descending modalities vary during the performance and mark also the way subjects look at what is considered as given and at what is supposed to be found.

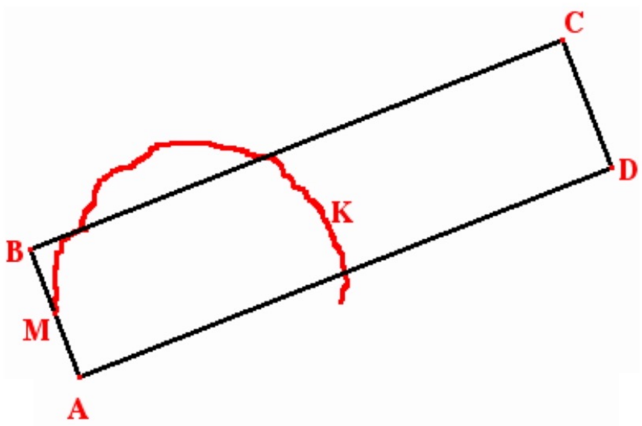
They constitute a delicate cognitive point, which has also a relevant didactic aspect. It is precisely in these two aspects that one can observe different dynamics between 'pencil & paper' and 'Cabri' environments.

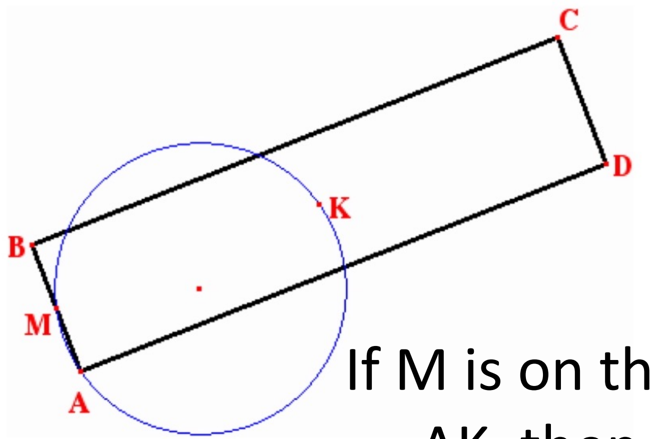
In both, the transition may be ruled by abductions; but while in the former the abductions are produced because of the ingenuity of the subjects, in Cabri the dragging process can mediate their productions.
Moreover, repeated switches between ascending/descending modalities support the evolution from perceptions towards a more theoretical frame: this evolution is marked by a kind of rhythm from ascending to descending modalities and back.

In the 2011 paper, M.A. and A.B.-F. entered more in the links between dragging, abductions and their nature. Typically, they described how abduction is linked to 'mantaining dragging':



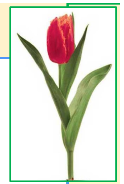






If M is on the circle of diameter AK, then ABCD is a rectangle

M.A. & A.B.-F. based on the elaboration of Peirce abduction given by L. Magnani and linked mantaining dragging to it.



Peirce:

Suppose I know that a certain bag is full of white beans.

Consider the following sentences:

A) these beans are white;

B) the beans in that bag are white;

C) these beans are from that bag.

A *deduction* is a concatenation of the form: B and C, hence A;

An *induction* would be: A and C, hence B;

An *abduction* is: A (FACT) and B (RULE), hence C (HYPOTHESIS).

An abduction produces an hypothesis.

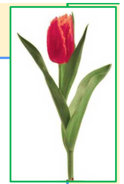
Peirce C S. (1960). *Collected Papers II, Elements of Logic*. Harvard: University Press

M.A. & A.B.-F. based on the elaboration of Peirce abduction given by L. Magnani and linked mantaining dragging to it.

Magnani:

“the process of inferring certain facts and/or laws and hypotheses that render some sentences plausible, that explain or discover some (eventually new) phenomenon or observation; it is the process of reasoning in which explanatory hypotheses are formed and evaluated.”

Magnani L. (2001). *Abduction, Reason, and Science: Processes of Discovery and Explanation.* New York: Kluwer Academic/Plenum Publishers.



A new interpretation of abductions is so given for Mantaining Dragging: the *hypothesis*, in Peirce's terms, is not to be considered as the product of the abduction. Instead, the product is the *rule:*

"if C is true then A is true $\sim$ the beans in that bag are white"

(a simple piece of information from the solver's bag of acquired knowledge). In the case of DGS dragging the rule is more complex and 'reverses' the way one is thinking at the beginning: this sounds as "were C true, then A would be true", and in its process of production we can identify two main components:

- the observation of two simultaneous occurrences, C and A;
- the choice of using maintaining dragging to search for a cause for the invariance of the interesting property A.

The first component (C and A) lies at the level of perception during the phenomenological experience, while the second (cause for the invariance) lies at a meta-level with respect to the first, and can give the solver awareness of the type of control, direct or indirect, exercised on each invariant, strengthening in this way the conjectured conditional link between C and A.

A similar refinement of abduction reasoning had been introduced from a logical standpoint in a paper by Arzarello et al. (1998), where first the approach of Lakatos to the logic of discovery was discussed and then analysed using the technicalities of the Natural Deduction (Prawitz, 1965) framework, stressing the new aspects that abductions can assume within the logic of discovery.

Prawitz,D. (1965). *Natural deduction: a theoretical study*. Stockholm, Goeteborg, Uppsala: Almqvist & Wicksell,

Arzarello, F., Andriano,V., Olivero,F., Robutti, O. (1998). Abduction and conjecturing in mathematics. In: Magnani, L., Nersessian N.J. & Thagard, P. (eds.), *Abduction and Scientific Discovery, Special Issue of Philosophica,* 61 (1):77-94.

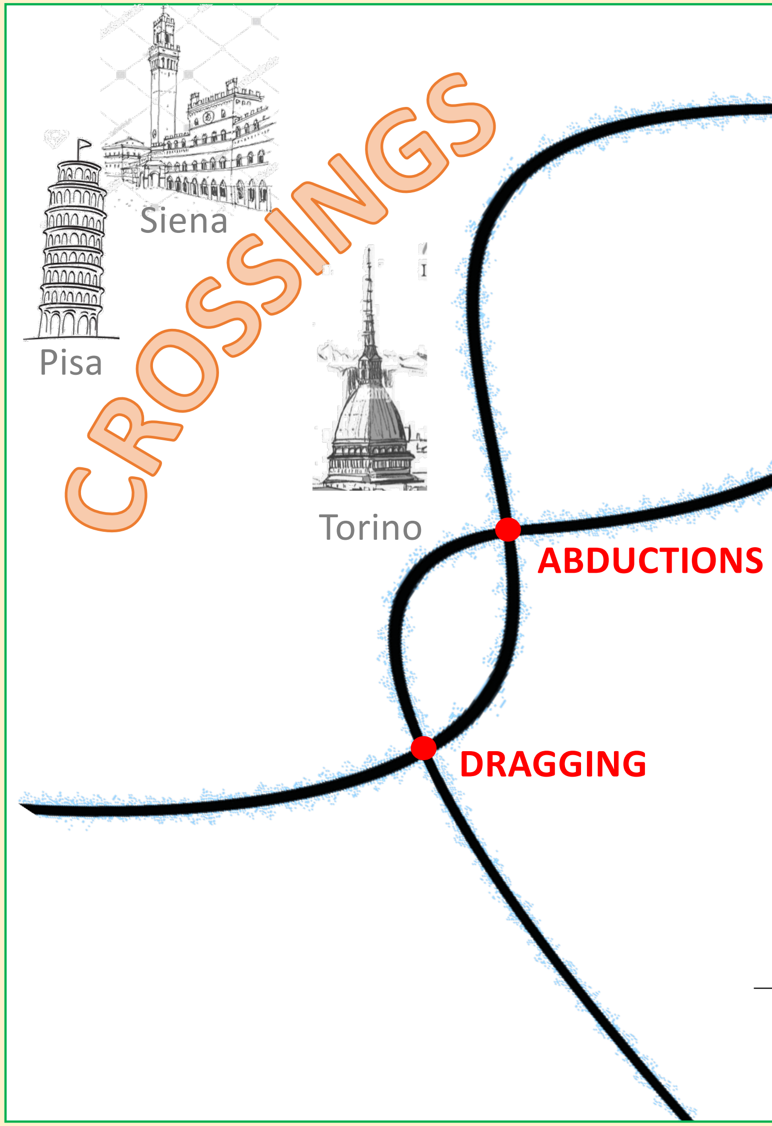
“ In our model (producing a conjecture and proving it) we can distinguish:
(i) a context, more precisely a fragment of a theory of reference, let us say P,
(ii) a surprising or interesting situation, E, worthwhile to be explained by a conjecture, namely a reason why E holds within P We can represent the resulting problematic situation in the following diagram

$$P \vdash (\ ?\ ) \dashrightarrow E$$

(iii) dynamic explorations (e.g. mantaining dragging), with *ascending control*, allow the subject to find such an hypothesis P', as a 'possible cause' of E within that context.

Then the *descending control* starts, possibly producing the final proof in the end, within a logic of discovering/proving, which result so deeply intertwined:

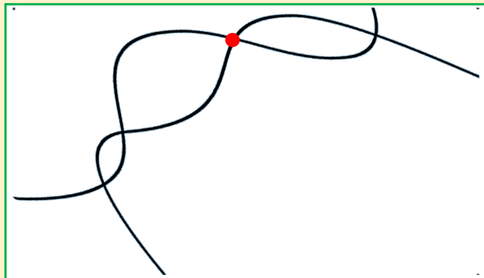
$$P \vdash P' \dashrightarrow E\ .$$ ” (p. 86)



Abduction is a significant kind of scientific reasoning, helpful in delineating the first principles of a new theory of science. It is situated at the crossroads of philosophy, epistemology, artificial intelligence, cognitive psychology, and logic.

(Magnani, *Preface*)

Magnani L. (2001). *Abduction, Reason, and Science: Processes of Discovery and Explanation.* Kluwer Academic



CROSSING 3
Indirect proofs and reverse reasoning

Antonini, S., Mariotti, M.A. (2006). Reasoning in an absurd world: difficulties with proof by contradiction. In *Proceedings of the 30th PME Conference.* pp.65-72.

Antonini, S., & Mariotti, M.A. (2007). Indirect proof: an interpreting model. *Proceedings of CERME 5*, Larnaca (Cyprus).

Antonini, S., Mariotti, M.A. (2008). Indirect proof: what is specific to this way of proving?. ZDM Mathematics Education 40, 401–412.

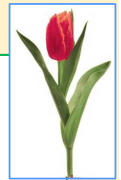
Antonini, S., & Mariotti, M.A. (2009). Abduction and the explanation of anomalies: the case of proof by contradiction. *Proceedings of CERME 6*, Lyon.

Arzarello, F., & Sabena, C. (2011). Semiotic and theoretic control in argumentation and proof activities. Educational Studies in Mathematics, 77(2), 189–206.

Arzarello, F., Soldano, C. (2019). Approaching Proof in the Classroom Through the Logic of Inquiry. In: Kaiser, G., Presmeg, N. (eds) Compendium for Early Career Researchers in Mathematics Education . ICME-13 Monographs. Springer, Cham.

“ In indirect proofs [...] something strange happens to the ‘reality’ of these objects. We begin the proof with a declaration that we are about to enter a **false, impossible world**, and all our subsequent efforts are directed towards ‘destroying’ this world, proving it is indeed false and impossible. We are thus involved in an act of mathematical destruction, not construction. Formally, we must be satisfied that the contradiction has indeed established the truth of the theorem (having falsified its negation), but psychologically, many questions remain unanswered. What have we really proved in the end? What about the beautiful constructions we built while living for a while in this false world? Are we to discard them completely? And what about the mental reality we have temporarily created?” (Leron, p. 323)

Leron, U. (1985). A Direct approach to indirect proofs. *ESM*, 16(3), 321–325



M.A.M. characterized the structure of indirect proofs in a very articulated way : “ Any mathematical theorem is characterised by a statement and a proof and that the relationship between statement and proof makes sense within a particular theoretical context, i.e. a system of shared principles and inference rules.
Historic - epistemological analysis highlights important aspects of this complex link and shows how it has evolved over the centuries. The fact that the reference theory often remains implicit leads one to forget or at least to underevaluate its role in the construction of the meaning of proof. For this reason its seems useful to refer to a 'mathematical theorem' as a system consisting of a statement, a proof and a reference theory”. (p. 29)

Mariotti, M.A. (2000). Introduction to proof: the mediation of a dynamic software environment. ESM (44). 25-53

Two types of indirect proofs

Let n be a natural number If n^2 is even then n is even	Let n be a natural number If n is odd then n^2 is odd	*proof by contraposition*
Let a and b be two real numbers If $ab = 0$ then $a = 0$ or $b = 0$	Let a and b be two real numbers If $ab = 0$, $a \neq 0$, and $b \neq 0$ then $1 = 0$	*proof by contradiction*

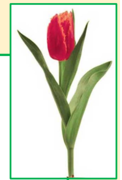
M.A.M & S.A., 2008. ZDM. p. 404

More precisely, in any theorem with indirect proof one M.A.M. and S.A. (CERME 5 paper) pointed out a complex structure, made of two theoretical levels, three statements, and three theorems:

(1) the sub-theorem (S*, C, T) consisting of the statement S* and a direct proof C based on a specific mathematical theory T (Algebra, Euclidean Geometry, and the like);

(2) a meta-theorem (MS, MP, MT), consisting of a meta- statement MS = S*→ S and a meta-proof MP based on a specific meta-theory MT (that usually coincides with classic logic);

(3) the principal theorem, consisting of the statement S and the indirect proof of S, based on a theoretical system consisting of both the theory T and the meta- theory MT’’. (p. 544)



M.A.M. & S.A. called indirect proof of S the pair consisting of the sub-theorem (S*, C, T) and the meta-theorem (MS, MP, MT); in symbols P = [(S*, C, T), (MS, MP, MT)].

In summary, an indirect proof consists of a couple of theorems belonging to two different logical levels: the level of the mathematical theory and the level of the logical theory.

Indirect argumentations Vs indirect proofs

By analyzing specific aspects of indirect proof, the model reveals its efficiency in identifying, analyzing and interpreting students' difficulties when dealing with proofs with an indirect structure.

The complexity of the logical structure of indirect proof, as highlighted by the model, can explain the difficulties met by the students.

But from the perspective of Cognitive Unity, it is reasonable to put forth the question whether similar difficulties can be found in the production of indirect argumentations.

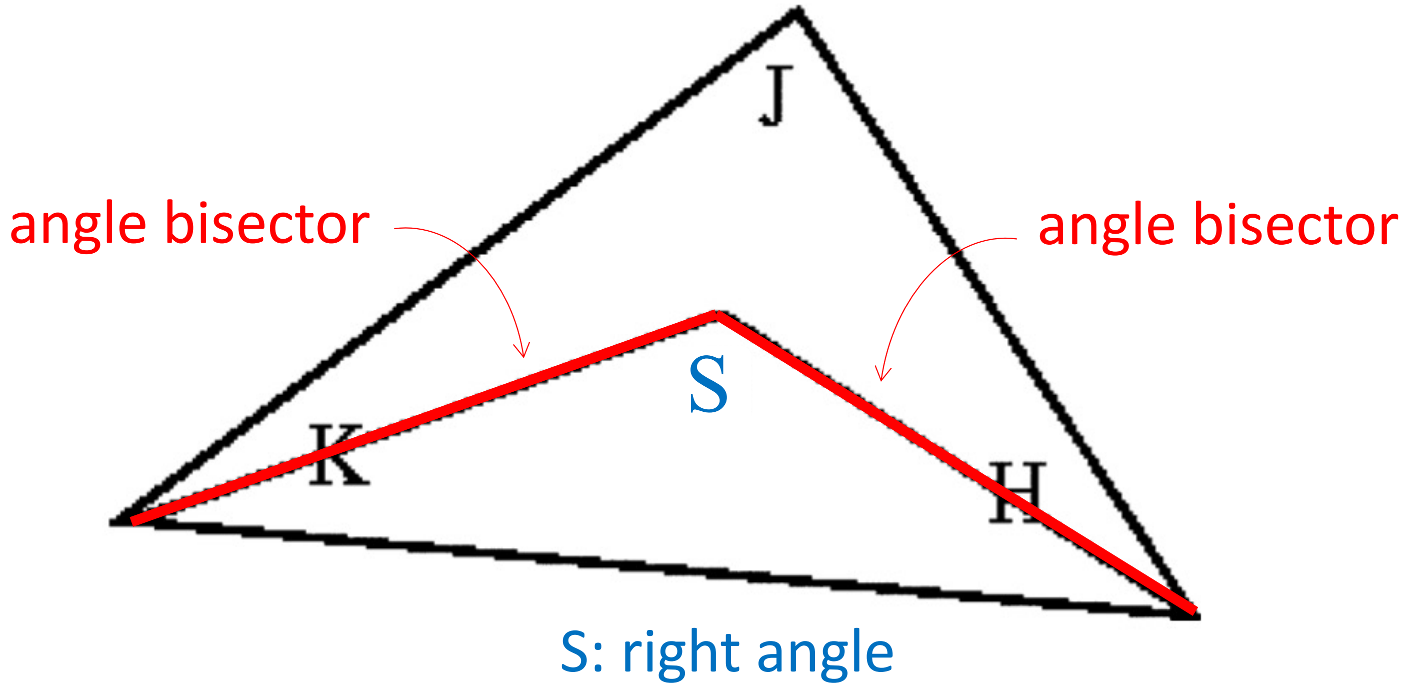
M.A. & S.A. were clever in pointing out important aspects of argumentations produced by students when facing the false impossible world linked to indirect proofs.

For this they studied indirect argumentations.

Let us see one of their nice examples: it shows the spontaneous production of an indirect argumentation supporting a conjecture, and some difficulties arising in the construction of the proof of the conjectured statement.

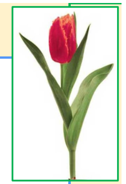
The analysis of the protocol, carried out in the frame of their model, highlights some difficulties in the application of the theory of Euclidean Geometry to an object that is geometrically inconsistent and how these difficulties can be overcome through argumentative processes.

The example for a ‘false, impossible world’



Abductions are again on the spot

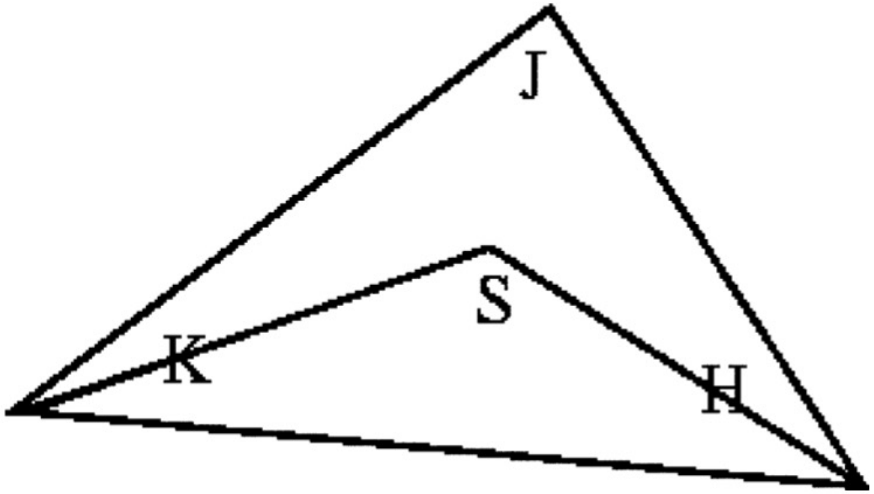
M.A.M & S.A., 2008. ZDM. p. 410 and ff.



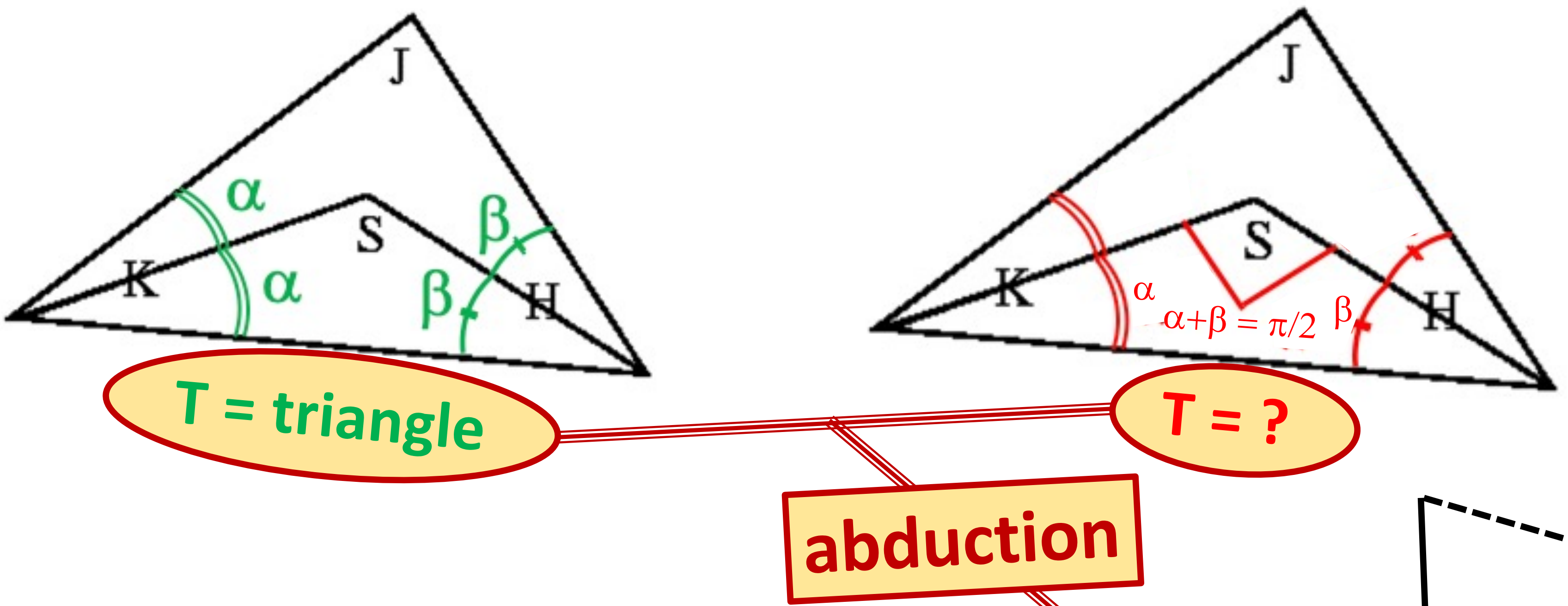
R: It would be impossible. Exactly, I would have with these two angles already 180°, that surely it is not a triangle.

[...]

R: We can exclude that the angle is right because it would become a quadrilateral.



The false world



If the angle is right then the sum of two angles of the triangle is 180°, then the triangle becomes a quadrilateral.

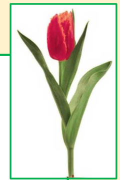
The fact that the angle S is a right angle is not excluded because of a contradiction.
Instead, it is excluded by the determination of a well-defined figure, as the consequence of the angle S being right.
An abduction has produced this reasoning: the final figure is a quadrilateral and this excludes the case of the triangle.
The arguments, by which it was possible to determine a figure and to show that it is not a triangle, are very convincing, and perhaps stronger than any argument based on a contradiction.
This may explain the immediate acceptability of this indirect argumentation.

“the activity of producing a conjecture can offer students the possibility of activating these processes and then of constructing a bridge to **overcome the gaps** that indirect proof seems to provoke.

On the contrary, without any conjecturing phase, some gaps could not be bridged or could require sacrifices and mental efforts that not all the students seem to be able to make”.

(p. 411)

M.A.M & S.A., 2008. ZDM.



“What about the beautiful constructions we built while living for a while in this false world? Are we to discard them completely? And what about the mental reality we have temporarily created?.“ (Leron 1985, p. 323)

In order that that they do not assume a merely temporary role, that their existence is not confined to a limited time in an absurd world destined to disappear, and that they are not only optical illusions, the objects in play (e.g. geometric figures) are transformed and replaced, and the same absurd and anomalous world where they lived could, under the new **abductive light**, regain meaning and appear completely normal. (S. Antonini, 2021)

The abductive light consists in the ‘reverse’ reasoning way embodied in students’ inverse arguments.
With this in mind, myself and C. Soldano analysed from a logical standpoint **gaps** like those provoked by indirect proofs, which had been pointed out by M.A.&S.A..

Along this ‘abductive line of thought’ we elaborated a game-theoretical approach to proofs, based on J. Hintikka Game Theory Logic.

It reverses the usual Traski definition of truth, and allows to frame the ‘reverse’ features of indirect argumentations, which, on their side, can trigger the production of conjectures.

---

van Ditmarsch, Sandu G. (Eds.) (2018).
Jaakko Hintikka on Knowledge and Game-Theoretical Semantics. Cham, Switzerland: Springer.

A double gap

between students' argumentative and proving processes is underlined by most of the literature (*) about the teaching of proof

The gap has both epistemological and cognitive features

G1. between what is empirically perceived as true and what is logically valid within a suitable theoretical framework.

G2. between a first arguing phase in students' productions and the proving phase.

EPISTEMOLOGICAL ASPECTS

COGNITIVE ASPECTS

A double gap

between students' argumentative and proving processes is underlined by most of the literature (*) about the teaching of proof

The gap has both epistemological and cognitive features

G1. between what is empirically perceived as true and what is logically valid within a suitable theoretical framework.

G2. between a first arguing phase in students' productions and the proving phase.

EPISTEMOLOGICAL ASPECTS

COGNITIVE ASPECTS

Logic of Inquiry (*)

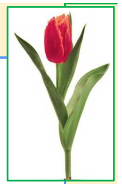
Abductions (°)

GAMIFICATION (#)

(*) Hintikka (1999)

(#) Arzarello & Soldano (2019)

(°) Peirce (1878); Magnani (2009)



A double gap

between students' argumentative and proving processes is underlined by most of the literature (*) about the teaching of proof

The gap has both epistemological and cognitive features

G1. between what is empirically perceived as true and what is logically valid within a suitable theoretical framework.

G2. between a first arguging phase in students' productions and the proving phase.

EPISTEMOLOGICAL ASPECTS

COGNITIVE ASPECTS

Logic of Inquiry (*)

Abductions (°)

GAMIFICATION (#)

TECHNOLOGY (@)

Robust /Soft constructions in DGE

(#) Arzarello & Soldano (2019). Approaching Proof in the Classroom Through the Logic of Inquiry. *ICME 13 Mon.*: 221-243

(@) Soldano, Luz, Arzarello, & Yerushalmy (2019). Technology-based inquiry in Geometry, *Ed. St. Math. 100(1),* 7-23

Logic of Inquiry

J. Hintikka (1929-2015) introduced the LI (or Game Theory Logic) into the logic of scientific discovery to overcome the static approach to reasoning represented by the usual mathematical logic. LI is an example of epistemic logic, which allows for rendering the two complementary processes of inquiring and deducing in a unitary frame.

LI model is characterized by:

i. the dialectic between questions and answers;
ii. the deep link with game theory;
iii. the functional interpretation of connectives and quantifiers.

van Ditmarsch, H. & Sandu, G. (Ed.s) (2018). *Jaakko Hintikka on Knowledge and Game-Theoretical Semantics*. Springer.

Rasmus, R., Symons, J., & Wang, Y. (2024). Epistemic Logic. In: Zalta, E.N. & Nodelman, U. (eds.), *The Stanford Encyclopedia of Philosophy* .

Abductions in the LI

The LI dialectic between **definitory rules** (framing the deductive steps) and the **strategic principles** (producing the inquiry steps) generates abductive forms of reasoning, not so far from the originary Pierce model (Hintikka, 1999). Magnani calls them manipulative abductions.

Magnani, L. (2009). *Abductive Cognition*. Springer. pp. 41-57

Manipulative Vs Theoretical abductions

MA are processes in which a hypothesis is formed and evaluated resorting to basically extra-theoretical behaviors. In our case this happens within technological tools (e.g. dragging activities): the game creates a kind of an ‘epistemic negotiation’ between the internal framework of the student and the external reality of the diagrams built with the digital tool because of the proposed game.

“Manipulative abduction happens when we are thinking through doing and not only, in a pragmatic sense, about doing” (Magnani, 2009, p. 46).

In this way, students’ actions, productions and communications with their dragging practices assume an epistemic and not a merely performative role, which is relevant for abductive reasoning.

The Natural Deduction model hinted before shows that in the Game Theory approach, there are essentially two strategies for attacking problems:

$$P \vdash (\ ?\ ) \rightarrow E \qquad\qquad P \,\&\, (\ ?\ ) \vdash E$$

In both cases *abduction* plays an essential role in reversing the course of thought.

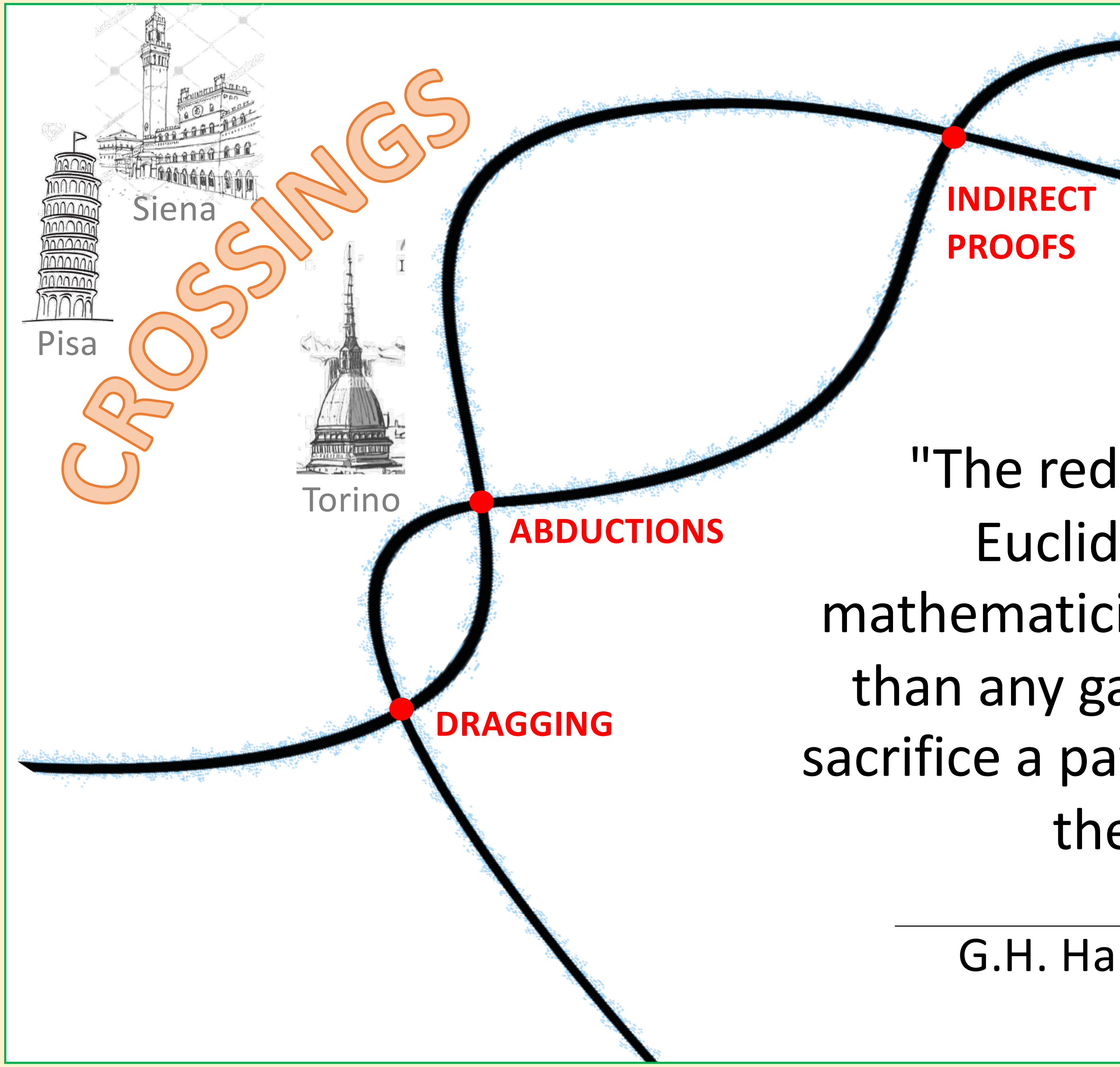
The use of counterexamples to find what is looked for in the game, seems to be at the origin of a course of thought, similar to the one found by M.A.M. & S.A. in their examples of reification of an impossible triangle through a quadrilateral.

In Arzarello & Sabena (2011), we have called such strategies: the *logic of not*.

I think it should be worthwhile to investigate this type of ‘abductive’ situations within both our frameworks.

Vive l'Abduction!

A new seminar of the Abduziosi
twenty years later?

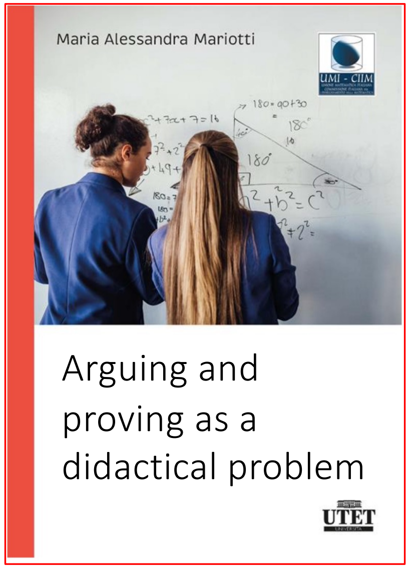


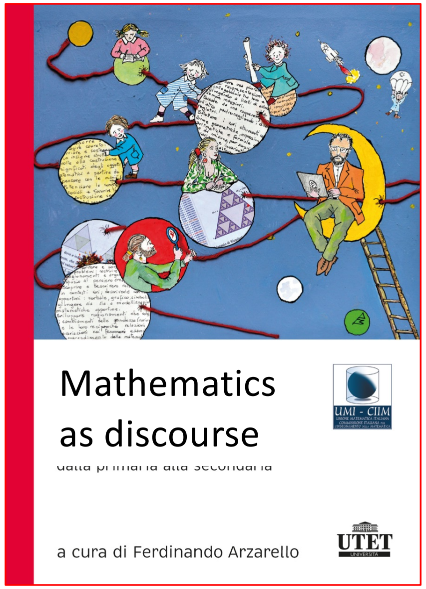
"The reductio ad absurdum, so beloved by Euclid, is one of the finest weapons of a mathematician. It is a gambit far more refined than any gambit in chess: a chess player may sacrifice a pawn or even some other piece, but the mathematician offers the game''

G.H. Hardy (1940). *A Mathematician's Apology.* Cambridge University Press. § 12.

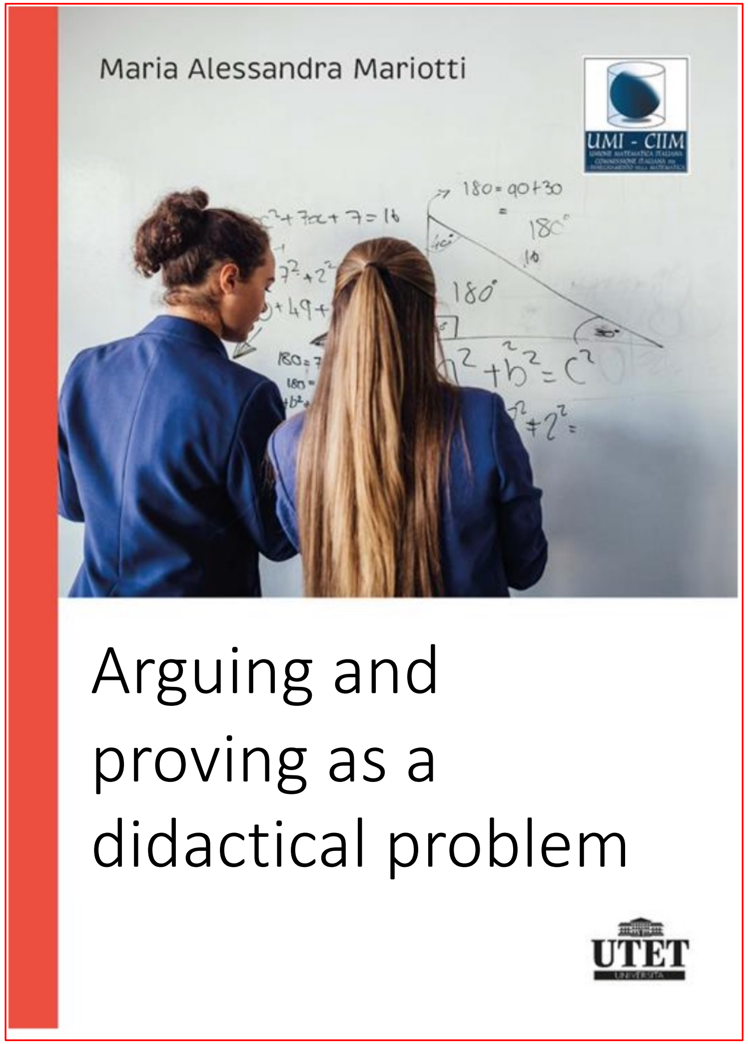
CROSSING 4
Educational books for teachers

Mariotti, M.A. (2022).
Argomentare e dimostrare come problema didattico.
Collana UMI-CIIM. Milano: UTET Università.



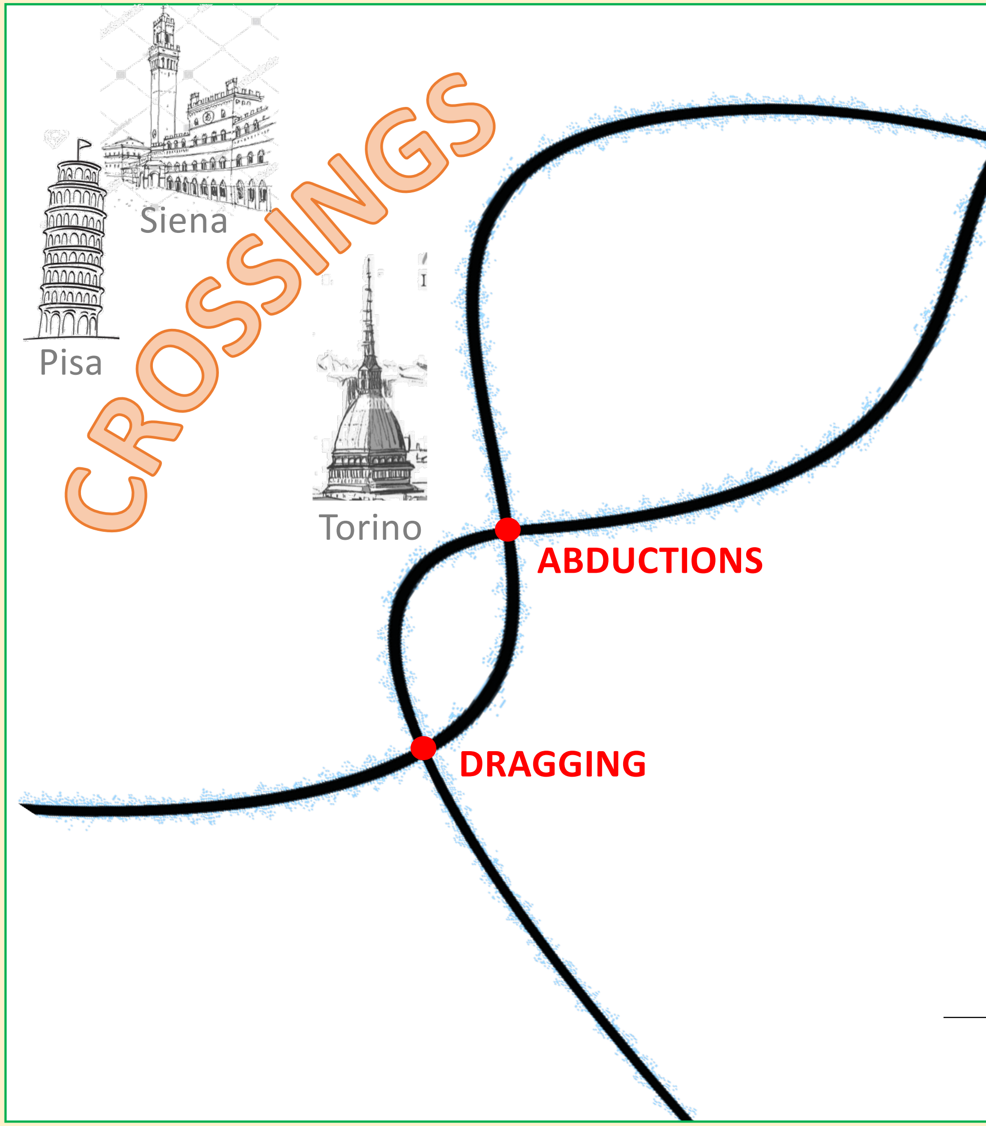


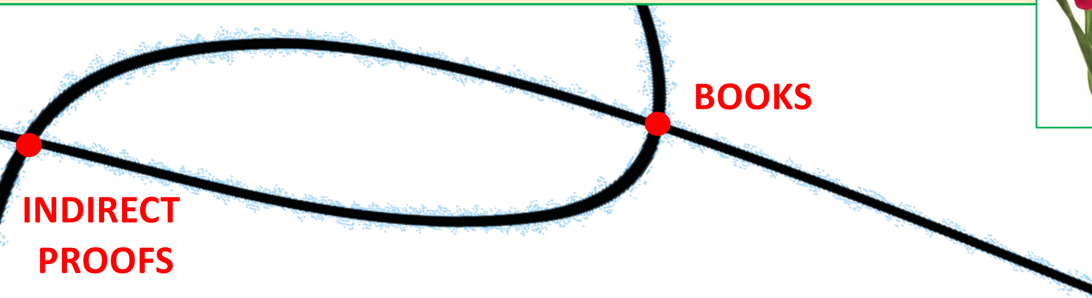
Arzarello, F. et al. (2024).
Matematica come discorso.
Collana UMI-CIIM. Milano: UTET Università.

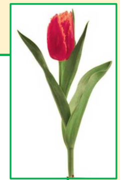

Maria Alessandra Mariotti
UMI - CIIM
Arguing and proving as a didactical problem
UTET

The first part of the volume (chapt.s I-IV) shows how the words “arguing [argomentare], explaining and proving are intertwined in a complex system of practices for which contiguity and distance must be kept in mind, in order to be able to plan and carry out activities in class that favor the development of argumentative skills” (p. 107). Practices are precisely the things whose names’ semantic contiguity and distance must be kept in mind for educational purposes.

The second part of the volume (chapters V-VIII) instead develops the didactic consequences of the complex theoretical framework woven in the first part.
Here recipes are avoided, but instead "some general principles illustrated by some examples are elaborated, principles that can be reinvested by the reader, for the personal planning of significant activities, aimed at developing the mathematical sense of argumentation".
In particular, she introduces the notion of cognitive unity/rupture between arguing and proving, illustrates the culture of theorems and that of ‘whys?’ in mathematical discussions, as well as the distinction between core and balance tasks, and the consequent role of the teacher in organizing and supporting them.

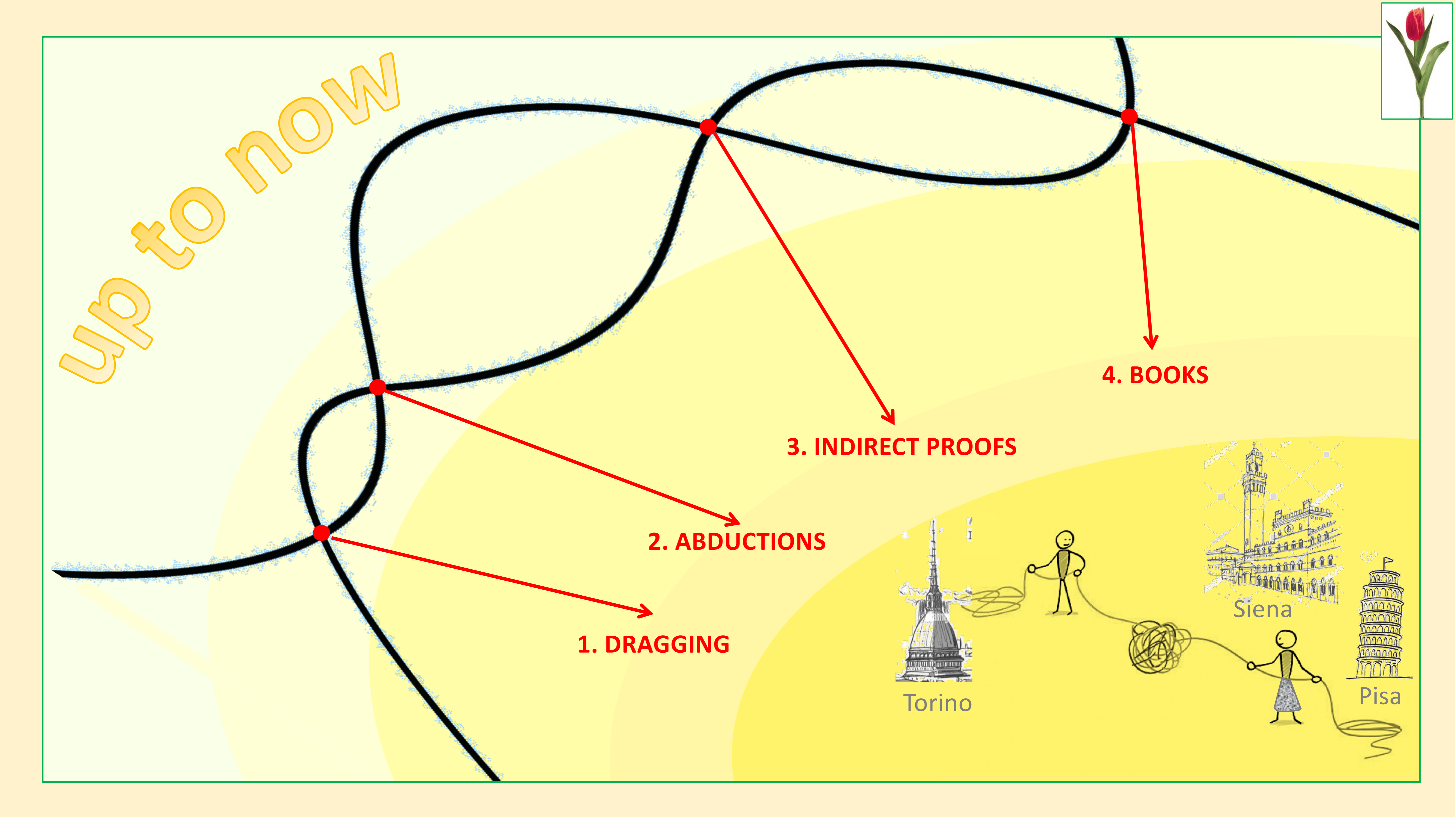


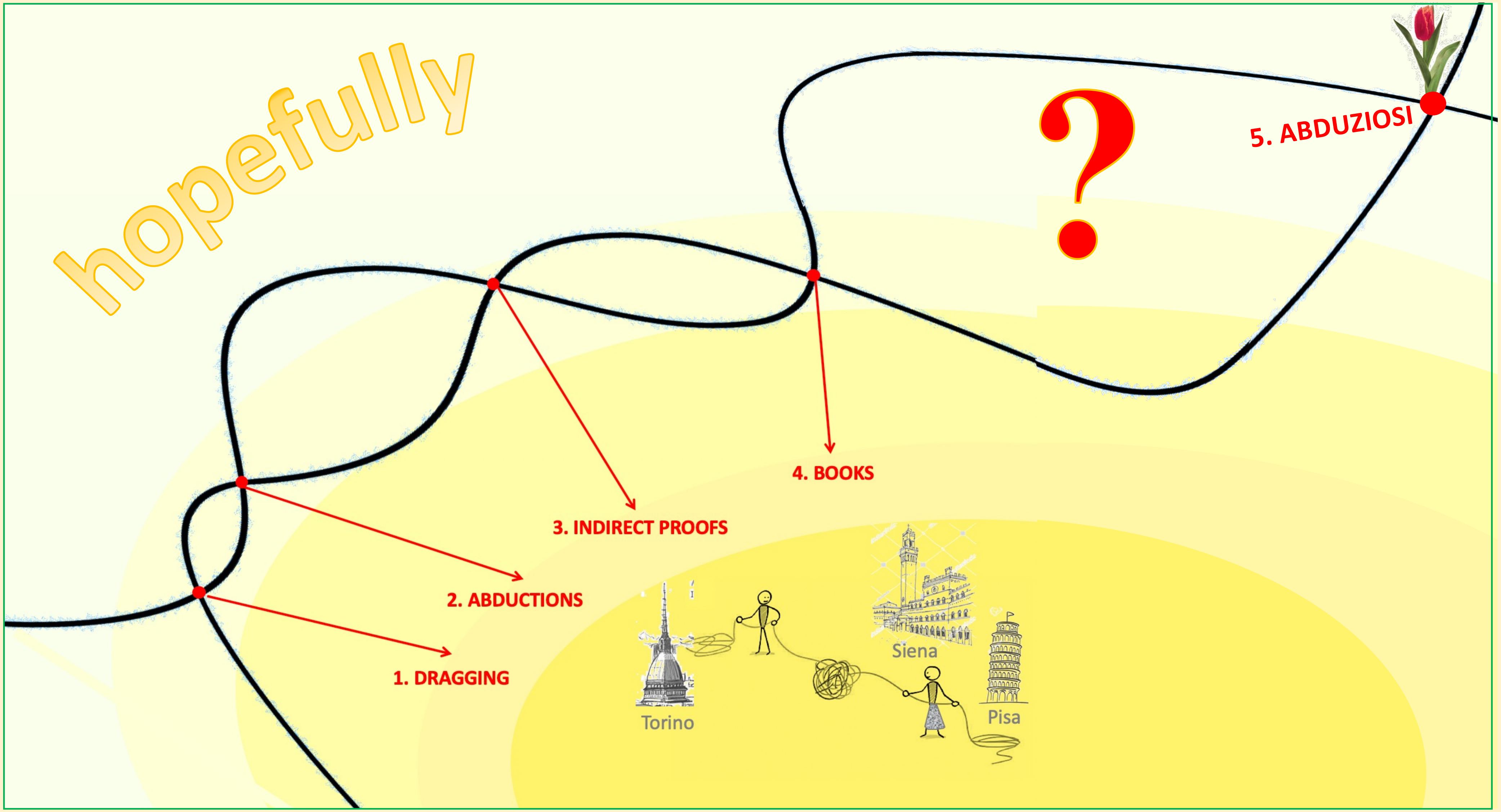




Dissemination is no longer seen as an ancillary activity, but as an integral part of the scientific mission. This paradigm shift has placed scientists in front of new responsibilities and challenges, partly redefining their professional role. In a democratic society, science thrives only if citizens perceive it as understandable, useful and not hostile.

Why Public Engagement Matter (2025*). American Association for the Advancement of Science.*


up to now
1. DRAGGING
2. ABDUCTIONS
3. INDIRECT PROOFS
4. BOOKS
Torino
Siena
Pisa


hopefully
5. ABDUZIOSI
?
4. BOOKS
3. INDIRECT PROOFS
2. ABDUCTIONS
1. DRAGGING
Siena
Torino
Pisa

Thank you for all,
Maria Alessandra!