UNFOLDING THE SYSTEMIC NATURE OF THEOREM

A tribute to Maria Alessandra Mariotti's epistemological clearsightedness

Nicolas Balacheff
Laboratoire d'informatique de Grenoble (LIG)
Univ. Grenoble Alpes, CNRS, Grenoble INP

Si nous devions exprimer notre évolution personnelle…

« If we had to express our personal evolution, we would say that we feel that our ideas on *mathematical proof* have changed in recent years, and that they are above all much more uncertain and confused...

it is precisely in our attempt to make them clearer that we have noticed how difficult it is to enclose the idea of *mathematical proof* in a discourse that would describe it. »

Original in French

NB: these words express a feeling which I share. It is the exact place where, in my research journey, I met Maria Alessandra. As it were, this feeling is contradictory to the last century common sense idea about mathematical proof – I.e. a formal proof expressed which associate formalism and natural language, a naïve formalism as Bourbaki would say.

Milestones on the way to disappearance

Theorem in the classical age (Legendre 1794):

Statement that* becomes obvious thanks to a *mathematical proof

Dimostrazione

Modern times (1960s)

Mathematical proof loses ground to ***deductive reasoning***

Contemporary evolution

2003	Justify/Prove	**Provide evidence for the validity** *of an action or the truth of a statement* by reference to mathematical results or properties; develop mathematical arguments to prove or disprove statements, given relevant information. (TIMSS 2003 p. 33)
2007	Justify	Provide **a justification** for the truth or the falsity of a statement by reference to mathematical results or properties (TIMSS 2007 p. 38)
2023	Justify	**Provide mathematical arguments** *to support a strategy or solution* (TIMSS 2023 p. 17)

Y/Our postulate:

You can't teach or learn math without mathematical proof

STATEMENT, **PROOF**, THEORY

NB: These words are the terms of the triplet which models the concept of Theorem as a system that Maria Alessandra coined.

I will come to this later. For the time being, I use it to structure the talk. So, let's first focus on Proof, the emblematical keyword of our research domain.

In the beginning is intuition…

Fishbein postulate: **The feeling of universal necessity** is a feeling of agreement, **an intuition** which is **congruent with the corresponding formal acceptance**

- intuition is a compact form of knowledge like perception
- like perception, it does not require extrinsic justification.
- it is a result of the involvement in solving problems raised by practical situations

Fishbein: Statement and proof must condense into an intuitive knowledge to become productive

Mariotti: Making **explicit the relationships** which are implicit at the intuitive level **constitutes the first step towards the construction of an argumentation**, which, *in the framework of a theory, can become a proof*

From practicioner to theoretician

From a theoretical point of view, a mathematical proof can be considered as completely independent of interpretation. But, from a cognitive point of view,…

… it is impossible to ignore the tension between

Conceptual proof and Deduction

→the source of tension rests in the relation between

- → the construction of knowledge & its systematization into a theoretical corpus
- → typical aspects of communication & typical aspects of knowledge as a cultural product

From practicioner to theoretician

The fundamental problems:

1. To resolve the possible conflict between the two functions of proof: **explaining and validating**

2. To achieve a flexible way of thinking that is able to move, in a casual and conscious manner

   From **the intuitive level** that of *truth* (in terms of the meanings of statements) to

   To **the formal level** of *validity* (in terms of the logical dependence between statements)

(STATEMENT, PROOF, THEORY**)**

=Theorem

Theorem as a unity of…

→ The need for semantically pregnant referents which the teacher and students can recognize and consider as unitary and homogeneous

A field of experience

→ The students produce statements through an intense activity, argumentation and justification, backing the *plausibility* of their choices.

Field of experience, statement

→ Then, students organize some of the justifications according to a logical chain

Field of experience, statement, argumentation

Continuity between the production of a conjecture and the construction of its proof

Theorems as unities of statement, proof and theory

Mathematical theorem as a system

In 1997…

Theoretically, a *definition* relates the new object to all the others, in such a way that a chain (system) of definitions is built up; *this system is an organic and coherent whole*.

The gap between a spontaneous defining process and a mathematical defining process concerns both the origin of the concepts and

their organisation within a theoretical system

Let’s focus on the idea of System…

“*The idea of Theorem you focus on, appeared to me so obvious, but at the same time it became crucial in experimental design of our long term teaching experiments.*”

Maria Alessandra Mariotti

Personal communication 07/03/25

Theorem as a system…

Field of experience, statement, argumentation

Theorems as
unities of statement, proof and theory

Theory as a system of shared principles and deduction rules is needed if we are to speak of proof in a mathematical sense.

Principles and deduction rules are so intimately interrelated so that what characterizes a mathematical Theorem…

is the mutual relationships among the three main components: a *statement*, its *proof* and the *theory* within which the proof makes sense

Theorem as the
system of statement and proof and theory

STATEMENT, PROOF, THEORY

Sentence

A sentence is a form of statement

The *utterance* of a **statement** requires

words & good command of rules of language

a **sentence**

appropriate to this communicative purpose must include

semantic adequacy and formal correctness

Arzarello three facets of mathematical formal correctness:

- *Facet 1* “concerns the form of **mathematical sentences**, as structured syntactical objects independently of their intertextual contexts”
- *Facet 2* “concerns the way mathematics is presented as a final product, in **a formalised language**, generally contrasted with that after which results are found by mathematicians”
- *Facet 3* “concerns the very notion of **logical consequence**”

Semiotic potential & mediation

Vygotskian postulate 1: students words and discourse forms is a progressive appropriation and reflexive use of ways of behaving with others and of others

The others… teacher and students
Words and **discourse** in mathematics

Ontological Constraint **Sociomathematical norm**

Signs act as instruments in a manner of tools in labor

signs expressing the relationship between the artifact and tasks
signs expressing the relationship between artifact and knowledge.

This double semiotic relationship [is] the semiotic potential of an artifact

The teacher uses the artifact as a tool of semiotic mediation

The teacher and… a situation

To foster giving meaning to the activity…

- ✓ the presence of **'concrete' and semantically pregnant referents**
- ✓ the **presence of semiotic mediation tools**;
- ✓ the **construction of an evolving student "internal context"**.

&

That is the creation of a shared space of reference

A field of experience:

"a sector of human culture which the teacher and students can recognize and consider as unitary and homogeneous."

Intentionality and cultural mediation

Vygotskian postulate 2: internalization is essentially social and directed by semiotic processes

- Core of social activities constituted by "**mathematical discussions**"
- Evolution of the field of experience through activities aimed to a social construction of knowledge.

In a mathematical discussion the whole class is collectively engaged in a mathematical discourse, usually launched by **the teacher**, explicitly formulating the theme.

Any artifact is a tool of semiotic mediation as long as it is *intentionally used* ***by the teacher*** to mediate a mathematical content through a designed didactical intervention.

the functional relationships between
the statement of a conjecture and the theory

STATEMENT, PROOF, **THEORY**

"Don't underestimate the notion of theory"

An intuition is a theory…

NB: In everyday conversations, a theory is an abstract idea that may not be applied in practice.

That sounds good in theory, but it won't work in real life

Fishbein: An intuition is a theory expressed in a particular representation using a model: a paradigm, an analogy, a diagram, a behavioural construct etc.

Intuition manifests itself in situations through

- Observable actions
- Semiotic systems
- Forms of control

NB: Intuition can be modelled by a conception with its domain of efficiency that grounds "intrinsic certainty"

The dialectic of intuition and formalisation

One engages in a search for a proof for good reasons, based one's **conceptions**, in support to the claim of the *truth of a statement*. ⇅ intuition formalisation

→ Formation of a **sentence** and a structured **organisation of arguments**.

→ Until *an explanation* is established *in the eyes of the problem-solver*

which could function as an **argumentation** for others

Hence the proposal to characterise a conjecture along the line of Mariotti's characterisation of theorem:

Conjecture={statement, argumentation, conception}

On the way to a theoretical attitude…

The following aspects can emerge in the course of an activity based on an artifact with a semiotic potential:

- The instrumental aspect that axioms and theorems have in respect to validating new elements of the theory
- The different status of hypothesis and thesis in a theorem became clearly expressed

Two aspects of the metatheoretical level should be explained and discussed in class:

- the acceptability of certain specific means of inference
- the fact that no other means of inference is acceptable.
- this combination of tools and theory

THEOREM
(SENTENCE, PROOF, THEORY)

Unfolding the complexity

When is an argumentation mathematical?

The student may or may not construct the intended meaning of **theorem**, or may or may not make the metatheoretical step to catch the meaning of **theory**.

Moreover, they might break **mathematical norms** …

However,

The linguistic appropriateness of the sentence

The explicitness of the knowledge references

The structural coherence of the discourse may be sufficient to keep students' argumentation within the range of **mathematical acceptability**

When is an argumentation mathematical?

Conjecture = {statement, argumentation, conception}

Valid statement

{sentence, mathematical argumentation, knowledge base, }

Theorem = {sentence, mathematical proof, theory}

… becomes an element of …

When is an argumentation mathematical?

NB: Alessandra Mariotti's conceptualisation of the theorem offers a foundational contribution to the didactical characterisation of proof. It allows us to move beyond the classical pitfall of attempting to "elementarise" mathematical proof — an approach that has proven inadequate ever since the early ambition to teach "mathematics for all".

It should be clear, transparent, convincing
first accepted a by the mathematics classroom
then confirmed by the teacher
although
students may not have used classical mathematical definitions or conventional symbols

Mathematical argumentation is a multimodal text in support of the claimed validity of a sentence

Knowledge base – explicit, established by and for the classroom community
Sentence – linguistically appropriate, semantically adequate, of a general stance
Argumentation – ethically minded, formally coherent, congruent to students conceptions, linking the sentence to the knowledge base

Generic examples & Thought experiments
are candidate patterns of mathematical argumentation

Becoming a socio-mathematical norm,
Mathematical argumentation shall frame
the elementary classroom as a mathematical society
-- although situated and provisory --

It prepares the student move from Practitioner to Theoretician (1990)

Thank you Maria-Alessandra

Thank you the math ed. Italian community

Thank you all!

Bibliography

Mariotti, M. A. (1998). Intuition and Proof: Reflecting on Fischbein's paper. *The Proof Newsletter,* http://www.lettredelapreuve.org

Mariotti, M. A. (2001). La preuve en mathématiques. *La lettre de la preuve*. http://www.lettredelapreuve.org http://www-didactique.imag.fr/preuve

Mariotti, M. A. (2024). La contribution des environnements technologiques à l'enseignement de la preuve. *Recherches en Didactique des Mathématiques*. Episciences. https://doi.org/10.46298/rdm.12906

Mariotti, M. A., & Fischbein, E. (1997). Defining in Classroom Activities. *Educational Studies in Mathematics*, 34, 219-148.

Mariotti, M. A., Bussi, M. G. B., Boero, P., Ferri, F., & Garuti, R. (1997). Approaching geometry theorems in contexts : From history and epistemology to cognition. In E. Pehkonen (Éd.), Proceedings of the 21st PME Conference (Vol. 1, p. 180-195). University of Helsinki.

Mariotti, M. A., & Pedemonte, B. (2019). Intuition and proof in the solution of conjecturing problems. *ZDM*, 51(5), 759-777.

Mariotti, M. A., Bussi, M. G. B., Boero, P., Ferri, F., & Garuti, R. (1997). Approaching geometry theorems in contexts: From history and epistemology to cognition. In E. Pehkonen (ed.), *Proceedings of the 21st PME Conference* (Vol. 1, p. 180-195). University of Helsinki

Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom. In L. D. English & D. Kirshner (Éds.), *Handbook of International Research in Mathematics Education* (2nd edition, p. 746-783). Routledge.

References

Arzarello, F. (2007). The proof in the 20th century. In P. Boero (ed.), *Theorems in School: From History, Epistemology and Cognition to Classroom Practice* (p. 43-63). BRILL.

Austin, J. L. (1950). In G. Longworth (ed.), Truth (The virtual issue n°1-2013). The Aristotelian Society. https://www.aristoteliansociety.org.uk/pdf/2013%20AS%20Virtual%20Issue.pdf

Balacheff N. (2024) Notes for a study of the didactic transposition of mathematical proof. *Philosophy of Mathematics Education Journal*. University of Exeter, United Kingdom.

Balacheff, N. (2024). Mathematical Argumentation, a Precursor Concept of Mathematical Proof. *Proceedings of the 14th International Congress on Mathematical Education*, 33-52.

Boero, P., Dapueto, C., Ferrari, P., Garuti, R., Lemut, E., Parenti, L., & Scali, E. (1995). Aspects of the mathematics-culture relationship in mathematics teaching-learning in compulsory school. *Proceedings of the 19th PME conference. PME-XIX, Recife.*

Fischbein, E. (1982). Intuition and proof. *For the Learning of Mathematics*, 3(2), 9-18.

Fischbein, H. (1987). *Intuition in Science and Mathematics: An Educational Approach*. Springer Science & Business Media.

Legendre, A.-M. (1794) *Éléments de géométrie*, avec des notes. https://gallica.bnf.fr/ark:/12148/bpt6k1521831j

Vygotskii, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, Mass.; London : Harvard University Press.