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Palindromic linearization and numerical solution of nonsymmetric algebraic $\top$-Riccati equations

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We consider the Nonsymmetric algebraic $\top$-Riccati equation ($\top$-NARE)

$$DX + X^\top A - X^\top BX + C = 0, \tag{1}$$

where X is the unknown matrix and $A, B, C, D \in \mathbb{R}^{n\times n}$ are the coefficients, while the superscript $\top$ denotes transposition. Equation (1) has been considered in [1], with applications to solving large-scale Dynamic Stochastic General Equilibrium models.

The $\top$-NARE takes its name from the nonsymmetric algebraic Riccati equation (NARE)

$$DX + XA - XBX + C = 0, \tag{2}$$

whose analysis and numerical solution has been of great interest in the literature in the last decades (see the books [2], [3]), and from the transposition $\top$ of the unknown X in (1), when it premultiplies a matrix coefficient. Indeed, recently, the $\top$ counter-part of classical linear matrix equations has been widely studied [4], [5].

A standard procedure when dealing with an algebraic Riccati equation is the "linearization" that relates its solutions to the invariant (deflating) subspaces of a matrix (pencil). This greatly improves the analysis and the numerical solution of the equation. For instance, Equation (2) is associated with the matrix

$$H = \begin{bmatrix} A & -B \\ -C & -D \end{bmatrix},$$

and X is a solution to (2) if and only if there exists an n-dimensional invariant subspace of H spanned by the columns of $\begin{bmatrix} I \\ X \end{bmatrix}$.

Here, we introduce a linearization for the $\top$-Riccati equation (1). More specifically, using the coefficients of the matrix equation, we construct a $\top$-palindromic pencil $\varphi(z) = M + zM^\top$ of size $2n$, that linearizes the equation: if $\varphi(z)$ is regular and if X is a solution to (1), then the columns of $\begin{bmatrix} I \\ X \end{bmatrix}$ span a deflating subspace of $\varphi(z)$ and also a kind of converse result holds.

This linearization, besides being interesting per se, opens the way to find solutions of $\top$-Riccati equations by relying on algorithms that compute bases of deflating subspaces of a matrix pencil, such as the QZ algorithm and the Doubling algorithm.

In our tests we show that the two latter algorithms are more efficient, in terms of computational cost and CPU time, than Newton's method, the reference algorithm in [1], keeping the same accuracy.

Another interesting feature of our linearization of the $\top$-NARE is that it captures the peculiar structure of the problem, and this structure can be exploited by applying the palindromic QZ algorithm [6], [7], a structured variant of the QZ. We develop a structured ordering procedure for the palindromic QZ algorithm that allows us to find the required basis and gives computational advantages, being superior, in terms of forward error, in some difficult problems.

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