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A recursive eigenspace computation for the Canonical Polyadic decomposition

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Tensors, or multiindexed arrays, play an important role in fields such as machine learning and signal processing. These higher-order generalizations of matrices allow for preservation of higher-order structure present in data, and low rank decompositions of tensors allow for compression of data and recovery of underlying information [12, 2, 3]. One of the most popular decompositions for tensors is the canonical polyadic decomposition (CPD) which expresses a tensor as a sum of rank one tensors.

An important feature of the CPD is that, with mild assumptions [6, 8, 11], the CPD of a low rank tensor is unique. It is this uniqueness that allows for extraction of component information from a signal tensor. Furthermore, for a tensor with a unique CPD, the CPD can often be found algebraically. Such an algebraic solution can typically be obtained with limited computation time, hence is often used as an initialization for optimization based methods when the tensor is noisy.

One of the most popular algorithms for algebraic computation of a CPD of a tensor is the generalized eigenvalue decomposition (GEVD) [4, 5, 10, 9]. The key idea behind the classical GEVD is that a factor matrix of a tensor may be obtained by computing the generalized eigenvectors of any subpencil of the tensor.

While in the noiseless setting GEVD can exactly recover a CPD, it has recently been shown that pencil based algorithms such as GEVD are unstable [1]. That is, the condition number for computing a generalized eigenvalue decomposition of a subpencil can be arbitrarily larger than the condition number [14] for computing the CPD of the tensor. In this talk we present an extension of the GEVD algorithm which significantly improves the stability of algebraic computation of the CPD.

The stability of computing generalized eigenvectors of a matrix pencil is heavily dependent on the separation between the generalized eigenvalues and the generalized eigenvectors of the pencil [13, 7]. In the case that the generalized eigenvalues and eigenvectors are well separated, a GEVD may be stably computed. However, when either a pair of generalized eigenvalues or generalized eigenvectors are near parallel, computation of the generalized eigenvectors becomes unstable. As such, the GEVD algorithm performs well when there is a subpencil for the tensor in which all generalized eigenvalues are well separated; however, GEVD runs into challenges if one is unable to find a subpencil in which all generalized eigenvalues are well separated.

It is not hard to show that the difficulty caused by poorly separated generalized eigenvalues necessarily occurs as tensor rank and dimensions increase. Indeed, taking a subpencil of a tensor is equivalent to projecting the columns of one of the tensor's factors to be vectors of length 2. The original columns lie in a vector space of dimension I where typically $I >> 2$. Roughly speaking, the stability of the original CPD computation is dependent on the separation of the columns of the original factor matrix, while the stability of the generalized eigenvalue decomposition used to compute the CPD is dependent on the separation between the projected columns. Of course, the separation between the columns can significantly decrease under a projection. This in turn causes instability for the GEVD algorithm.

We address this fundamental issue by using many different pencils to compute the CPD. Intuitively, this allows us to consider many projections of the original factor columns, and allows us to take advantage of the fact that given clusters of columns will be better separated under some projections than others. More precisely rather than using a single pencil and computing all of its generalized eigenvectors, we use many different pencils and in each pencil compute generalized eigenspaces corresponding to sufficiently well separated generalized eigenvalues. The generalized eigenspaces we compute are then used to decompose the tensor in question as a sum of tensors with reduced rank. This is done in a way so that the CPD of the original tensor can be recovered by computing the CPDs of the summand tensors. Though the resulting "generalized eigenspace decomposition" is still fundamentally pencil based, it is significantly more robust to noise than the classical GEVD.

We will present a detailed explanation of the generalized eigenspace decomposition algorithm, and we will compare the performance in terms of accuracy and computational time of the generalized eigenspace decomposition to GEVD. In addition, we will examine stability of the generalized eigenspace decomposition both empirically and theoretically.

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