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Computing the condition number of tensor decompositions through Tucker compression

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In this talk, we investigate the condition number of *structured block term decompositions* [1], which are a general class of tensor decompositions encompassing the tensor rank decomposition, the block term decomposition, and sums of tensor train products. These decompositions express a tensor $\mathcal{A}$ as a sum of R "simple" terms $\mathcal{A} = \sum_{r=1}^{R} \mathcal{A}_r$. Specifically, the summand $\mathcal{A}_r$ can be expressed as a multilinear product $(U_1, \ldots, U_D) \cdot \mathcal{C}$ where $U_1, \ldots, U_D$ are full-rank matrices and $\mathcal{C}$ lies on a manifold $\mathcal{M}_r \subseteq \mathbb{R}^{l_1 \times \cdots \times l_D}$ that satisfies two assumptions:

1. Every $\mathcal{X} \in \mathcal{M}_r$ has multilinear rank $(l_1, \ldots, l_D)$.
2. $\mathcal{M}_r$ is invariant under changes of basis.

In practice, the given tensor $\mathcal{A}$ is almost always corrupted by noise, so it is essential to quantify how sensitive the summands $\mathcal{A}_1, \ldots, \mathcal{A}_R$ are to perturbations of $\mathcal{A}$. We show how Rice's condition number [2] can be applied to this decomposition. If $\widetilde{\mathcal{A}}$ is sufficiently close to $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ has a structured block term decomposition $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}_1 + \cdots + \widetilde{\mathcal{A}}_R$, then the sensitivity is bounded by

$$\|(\widetilde{\mathcal{A}}_1 - \mathcal{A}_1, \ldots, \widetilde{\mathcal{A}}_R - \mathcal{A}_R)\| \lesssim \kappa \|\widetilde{\mathcal{A}} - \mathcal{A}\|$$

where κ is the condition number and $\|\cdot\|$ is the Frobenius norm.

The computation of structured block term decompositions is often sped up by applying a dimensionality technique known as Tucker compression. That is, one expresses $\mathcal{A} = (Q_1, \ldots, Q_D) \cdot \mathcal{G}$ where $Q_1, \ldots, Q_D$ are matrices with orthonormal columns and $\mathcal{G}$ is a tensor with much smaller dimensions than $\mathcal{A}$. Then, $\mathcal{G}$ is decomposed as $\mathcal{G} = \sum_{r=1}^{R} \mathcal{G}_r$, which corresponds to a structured block term decomposition $\mathcal{A} = \sum_{r=1}^{R} (Q_1, \ldots, Q_D) \cdot \mathcal{G}_r$.

Since $\mathcal{G}$ has fewer possible perturbations than $\mathcal{A}$, one would expect the condition number of its decomposition to be smaller than that of $\mathcal{A}$'s decomposition. However, our main result is that the two condition numbers are equal. This is in contrast to other problems, where the condition number of the structured problem is much lower than that of the unstructured one [3].

Our result implies an algorithm to compute the condition number of the decomposition of $\mathcal{A}$ based on Tucker compression. This algorithm can reach a speedup of over four orders of magnitude relative to the state of the art in practical cases, so that it is now possible to compute the condition numbers of decompositions of large tensors.

References

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[3] B. Arslan, V. Noferini, F. Tisseur, *The structured condition number of a differentiable map between matrix manifolds, with applications*, SIAM J. Matrix Anal. Appl., (2019), pp. 774-799