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Hierarchical adaptive low-rank format with applications to discretized PDEs

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Low-rank based data compression can sometimes lead to a dramatic acceleration of numerical simulations. A striking example is the solution of time dependent elliptic PDEs of the form:

$$\begin{cases} \frac{\partial u}{\partial t} = Lu + f(t, u, \nabla u) & (x, y) \in \Omega, \quad t \in [0, T_{\max}] \\ u(x, y, 0) = u_0(x, y) \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a rectangular domain, L is a linear differential operator, f is nonlinear and (1) is coupled with appropriate boundary conditions in space. When the source term and the solution are smooth, their (structured) discretizations lead to matrices that allow for excellent low-rank approximations. Under suitable assumptions on the differential operator, one can recast the corresponding discretized PDE as a matrix equation [1, 2]. In turn, this yields the possibility to facilitate efficient algorithms for matrix equations with low-rank right-hand side [3, 4]. However, in many situations of interest the smoothness property is not present in the whole domain. A typical instance are solutions that feature singularities along curves, while being highly regular elsewhere. This renders a global low-rank approximation ineffective. During the last decades, there has been significant effort in developing hierarchical low-rank formats that apply low-rank approximation only locally [5]. These formats recursively partition the matrix into blocks that are either represented as a low-rank matrix or are sufficiently small to be stored as a dense matrix. These techniques are usually applied in the context of operators with a discretization known to feature low-rank off-diagonal blocks, such as integral operators with singular kernel.

We propose a new format that automatically adapts the choice of the hierarchical partitioning and the location of the low-rank blocks without requiring the use of an admissibility criterion. The admissibility is decided on the fly by the success or failure of low-rank approximation techniques. We call this format *Hierarchical Adaptive Low-Rank (HALR)* matrices.

We discuss techniques for the efficient adaptation of the structure in case of moving singularities, with the aim of tracking the time-evolution of the solution of (1); the numerical tests demonstrate that the proposed techniques can effectively detect changes in the structure, and ensure the desired level of accuracy. We develop efficient Lyapunov and Sylvester solvers for matrix equations with HALR right-hand-side and *Hierarchically off-diagonal low-rank* (HODLR) coefficients. Several numerical experiments demonstrate both the effectiveness and the flexibility of the approach.

References

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