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Solving the Discrete Euler-Arnold Equations for the Generalized Rigid Body Motion

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In [4], Moser and Veselov proposed the following equations to discretize the classical Euler-Arnold differential equations for the motion of a rigid body:

$$\begin{aligned} M_{k+1} &= \omega_k M_k \omega_k^T \\ M_k &= \omega_k^T J - J\omega_k, \end{aligned} \tag{1}$$

where M_k is the angular momentum with respect to the body (here represented by a skew-symmetric matrix), J is the inertia matrix (symmetric positive definite), and ω_k (orthogonal matrix) is the angular velocity. Rigid body equations arise in several applications, e.g., celestial mechanics, molecular dynamics, mechanical robotics and flight control, where they are used in particular to understand the body–body interactions of particles like planets, atoms and molecules.

The main challenge of solving (1) is to find an orthogonal matrix ω_k in the second equation, by assuming that J and M_k are given. Mathematically, the problem may be formulated as finding a special orthogonal matrix X ($X^TX = I$, $\det(X) = 1$) such that

$$XJ - JX^T = M, \tag{2}$$

where J is a given symmetric positive definite matrix, and M is a known skew-symmetric matrix. The matrix equation (2) was firstly investigated in [4], where the authors based their developments on factorizations of certain matrix polynomials. A different approach, but computationally more efficient, was provided later in [2], where the authors noted that (2) can be connected with a certain algebraic Riccati equation and, in turn, with the *Hamiltonian* matrix

$$\mathcal{H} = \begin{bmatrix} M/2 & I \\ M^2/4 + J^2 & M/2 \end{bmatrix}. \tag{3}$$

It is stated in [2] that (2) has a solution $X \in \mathcal{SO}(n)$ (the special orthogonal or rotation group of order n) if and only if the size of the Jordan blocks associated to the pure imaginary eigenvalues of $\mathcal{H}$ (if any) is even. The existing algorithms for solving (2) only work when $\mathcal{H}$ does not admit any pure imaginary eigenvalue. Moreover, the algorithms based on solving the associated algebraic Riccati equation require the strong condition that the matrix $M^2/4 + J^2$ must be symmetric positive definite. These issues have motivated us to investigate methods whose applicability does not require those restrictive conditions.

The problem of finding a special orthogonal solution X in (2) can be formulated as an optimization problem in the following way:

$$\min_{X \in \mathcal{SO}(n)} \left\| XJ - JX^T - M \right\|_F^2, \tag{4}$$

where $\|.\|_F$ denotes the Frobenius norm. Techniques from Riemannian geometry to solve optimization problems with orthogonal constraints have attracted the interest of many researchers in the last decades; see [1, 3], and the references therein. An essential feature of those techniques is that they allow the transformation of a constrained optimization problem into an unconstrained one. Since the set of orthogonal matrices is a manifold and provided that the objective function satisfies some smoothness requirements, we can make available tools such as Euclidean gradients, Riemannian gradients, retractions, and geodesics.

In this talk, we propose two iterative methods for solving (4). They evolve on the orthogonal manifold and belong to the family of line search methods on matrix manifolds described in [1, Ch. 4]. They are constraint-preserving, in the sense that, starting with a matrix $X_0 \in \mathcal{SO}(n)$, all the iterates X_k also stay in $\mathcal{SO}(n)$. The first one splits the orthogonal constraints using the Bregman method, whereas the second method is of steepest-descent type, based on a Cayley-transformation to preserve the constraints and on a Barzilai-Borwein step size. A set of numerical experiments are carried out to compare the performance of the proposed algorithms, suggesting that the first algorithm has the best performance in terms of accuracy and number of iterations. An essential advantage of these two iterative methods is that they work even when the conditions for applicability of the direct methods available in the literature are not satisfied. That is, they allow the computation of special orthogonal solutions, even when $M^2/4 + J^2$ is not symmetric positive definite. Those iterative algorithms may also be used in problems where $\mathcal{H}$ has purely imaginary eigenvalues associated with Jordan blocks of even size, but, as will be illustrated with experiments, the convergence may slow down.

References

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