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Solving large-scale Riccati equations with indefinite quadratic terms

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Algebraic Riccati equations with indefinite quadratic terms of the form

$$A^{\mathsf{T}} X E + E^{\mathsf{T}} X A + E^{\mathsf{T}} X \left(B_1 B_1^{\mathsf{T}} - B_2 B_2^{\mathsf{T}} \right) X E + C^{\mathsf{T}} C = 0, \tag{1}$$

with $A, E \in \mathbb{R}^{n \times n}$, $B_1 \in \mathbb{R}^{n \times m_1}$, $B_2 \in \mathbb{R}^{n \times m_2}$, $C \in \mathbb{R}^{p \times n}$ and E invertible, play an important role in applications related to robust controller design and differential games; see, e.g., [5, 7].

A particular interest lies in the existence and computation of a symmetric positive semi-definite, stabilizing solution $X_\infty \in \mathbb{R}^{n \times n}$ of (1). In other words, we want to compute an X_∞ that solves (1), that is symmetric positive semi-definite, and that ensures that the eigenvalues of the matrix pencil $\lambda E - (A + B_1 B_1^{\mathsf{T}} - B_2 B_2^{\mathsf{T}}) X_\infty E$ all lie in the left open half-plane.

While there are some established approaches to that in the case of small-scale dense coefficient matrices [1, 6, 8], there is no approach available to compute solutions in the large-scale sparse setting. In our work, we propose an extension of the iterative procedure developed in [6] to efficiently compute the requested solution of (1) in the large-scale sparse case via low-rank approximations such that $Z_\infty Z_\infty^{\mathsf{T}} \approx X_\infty$, with $Z_\infty \in \mathbb{R}^{n \times r}$ and $r \ll n$. The approach is based on considering the Riccati operator

$$\mathcal{R}(X) := A^{\mathsf{T}} X E + E^{\mathsf{T}} X A + E^{\mathsf{T}} X (B_1 B_1^{\mathsf{T}} - B_2 B_2^{\mathsf{T}}) X E + C_1^{\mathsf{T}} C_1.$$

For two symmetric matrices $X_1 = X_1^{\mathsf{T}}$ and $X_2 = X_2^{\mathsf{T}}$, one can show that

$$\mathcal{R}(X_1 + X_2) = \mathcal{R}(X_1) + \widetilde{A}^{\mathsf{T}} X_2 E + E^{\mathsf{T}} X_2 \widetilde{A} + E^{\mathsf{T}} X_2 (B_1 B_1^{\mathsf{T}} - B_2 B_2^{\mathsf{T}}) X_2 E$$

holds, where $\widetilde{A} := A + (B_1 B_1^{\mathsf{T}} - B_2 B_2^{\mathsf{T}}) X_1 E$. In the case that X_2 is a solution to the algebraic Riccati equation with negative semi-definite quadratic term

$$0 = \mathcal{R}(X_1) + \widetilde{A}^{\mathsf{T}} X_2 E + E^{\mathsf{T}} X_2 \widetilde{A} - E^{\mathsf{T}} X_2 B_2 B_2^{\mathsf{T}} X_2 E, \tag{2}$$

the residual reads

$$\mathcal{R}(X_1 + X_2) = E^{\mathsf{T}} X_2 B_1 B_1^{\mathsf{T}} X_2 E.$$

This leads to an iterative procedure, where in each step a Riccati equation of the form (2) needs to be solved. Using low-rank approximations for the intermediate Riccati equations (2) and some clever formulations of the iteration matrices allows the use of classical large-scale sparse solvers for (2), like the ones described, e.g., in [2, 3, 10]. Together with a reformulation of the overall iteration approach in [6], this leads to our new *low-rank Riccati iteration (LR-RI)* method.

Implementations of this new approach are available in [4] for dense systems and in [9] for the large-scale sparse case.

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