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Solving power series matrix equations encountered in stochastic processes by means of fixed point iterations

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We consider the problem of computing the minimal nonnegative solution G of the nonlinear matrix equation

$$X = \sum_{i=-1}^{\infty} A_i X^{i+1}$$

where A_i, for $i \geq -1$, are nonnegative square matrices such that $\sum_{i=-1}^{\infty} A_i$ is stochastic. This equation is fundamental in the analysis of M/G/1-type Markov chains which model a large variety of queuing problems in applied probability. The solution G, besides having an interesting probabilistic interpretation, provides an effective tool for representing the invariant probability measure of the chain. Classical fixed point iterations have been designed and analyzed in the literature and are customarily used for the computation of the matrix G. Their convergence can slow down when the problem is close to be null recurrent.

In this talk we introduce a new family of fixed point iterations for the numerical computation of G, that includes the classical iterations. The idea relies on rewriting the original equation as a polynomial matrix equation of degree $q+1$ of the form

$$X = B_{-1}(X) + B_0(X)X + B_1(X)X^2 + \cdots + B_q(X)X^{q+1}, \tag{1}$$

where the coefficients $B_i(X)$ are suitable functions of X. The sequence $\{X_k\}_{k\geq 0}$ generated by the fixed-point iteration, given X_0, is such that the value of X_{k+1} is the minimal nonnegative solution of the polynomial equation (1) with coefficients $B_i(X_k)$. The value of q and the choice of $B_i(X)$ characterize the specific fixed-point iteration in the class.

We show that the classical iterations are obtained for $q = 0$. Moreover, by means of a general convergence analysis, for any $q > 0$ we determine the coefficients $B_i(X)$ that maximize the convergence speed. As a result, we obtain new fixed-point iterations which are much faster than the classical ones. Numerical experiments confirm the effectiveness of our extension.

References

[1] D.A. Bini, G. Latouche, B. Meini, *A family of fast fixed point iterations for M/G/1 Markov chains* IMA J. of Numerical Analysis 2021.