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Optimal Constant for Generalized Diagonal Update Method

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The diagonal update method can be used in the Bernoulli's method to solve a quadratic matrix equation $AX^2 + BX + C = 0$, and it has better results on iteration number and time than the pure Bernoulli's method [1]. In this talk, we suggest the optimal constant which extends the sufficient condition to use the diagonal update method and guarantees the monotone convergence. Moreover, with some numerical experiments, we also compare the number of iterations defined by the generalized diagonal update method and the pure Bernoulli's method. Furthermore, we show that this generalized diagonal update method is useful to solve matrix equations with the form $AX^2 + \epsilon BX + C = 0$.

In detail, we consider the following quadratic matrix equation

$$Q_1(X) = AX^2 + BX + C = 0 \tag{1}$$

where

$$\begin{aligned} &A \in \mathbb{R}^{n\times n} \text{ is a diagonal matrix with positive diagonal elements,}\\ &B \in \mathbb{R}^{n\times n} \text{ is a nonsingular } M\text{-matrix,}\\ &C \in \mathbb{R}^{n\times n} \text{ is an } M\text{-matrix such that } B^{-1}C \text{ is nonnegative.}\end{aligned}$$

The equation (1) was motivated by a quadratic eigenvalue problem arising from an overdamped vibrating system [4]. In order to improve the pure Bernoulli's method in [2] and the diagonal update method in [1], we suggest the optimal constant $\gamma^* := \min\{\text{real}(\text{eig}(B - C)), 2\}$ and the generalized diagonal update skills:

$$\mathcal{G}_\gamma(X) = -\,(B + X - (\gamma - 1)\delta_X I)^{-1}(C + (\gamma - 1)\delta_X X), \tag{2}$$

$$\mathcal{H}_\gamma(X) = -\,(B - \gamma\delta_X I)^{-1}(X^2 + \gamma\delta_X X + C), \tag{3}$$

where $\delta_X = \min\{1, \min\{|\text{diag}(X)|\}\}$ and $1 \le \gamma < \gamma^*$. When $B - C - I$ is a nonsingular M-matrix, we can prove that both Berboulli's iterations defined by (2) and (3) with $X_0 = 0$ converge to the primary solvent X^*, by using some properties of M-matrices which are in [3].

Furthermore, we consider a numerical example with the following $m \times m$ coefficient matrices:

$$A = I, \;\; B = \epsilon \begin{bmatrix} 20 & -10 & & & & \\ -10 & 30 & -10 & & & \\ & -10 & 30 & -10 & & \\ & & -10 & \ddots & \ddots & \\ & & & \ddots & 30 & -10 \\ & & & & -10 & 20 \end{bmatrix}, C = \begin{bmatrix} 15 & -5 & & & & \\ -5 & 15 & -5 & & & \\ & -5 & 15 & -5 & & \\ & & -5 & \ddots & \ddots & \\ & & & \ddots & 15 & -5 \\ & & & & -5 & 15 \end{bmatrix}$$

which are motivated by [1] and [5], where $\epsilon \in \mathbb{R}$. We used the following algorithms:

$$\begin{cases} X_0 = 0, \;\; \delta_X = \min\{1, \min\{|\mathrm{diag}(X)|\}\}, \;\; \gamma = \min\{\mathrm{real}(\mathrm{eig}(B - C)), 2\} - 0.0001 \\ X_{i+1} = -(B + X_i)^{-1} C, & \text{(BI1)} \\ X_{i+1} = -(B + X_i - (\gamma - 1)\delta_{X_i} I)^{-1}(C + (\gamma - 1)\delta_{X_i} X_i). & \text{(BI1-OC)} \end{cases}$$

$$\begin{cases} X_0 = 0, \;\; \delta_X = \min\{1, \min\{|\mathrm{diag}(X)|\}\}, \;\; \gamma = \min\{\mathrm{real}(\mathrm{eig}(B - C)), 2\} - 0.0001 \\ X_{i+1} = -B^{-1}(X_i^2 + C), & \text{(BI2)} \\ X_{i+1} = -(B - \gamma\delta_{X_i} I)^{-1}(X_i^2 + \gamma\delta_{X_i} X_i + C). & \text{(BI2-OC)} \end{cases}$$

When $\epsilon = 0.95$, we can not use the original diagonal update method in [1], because $B - C - 2I$ is not a nonsingular M-matrix. Fortunately, if we use the generalized diagonal update method with $\gamma = 1.8683$, we have the good results as follows:

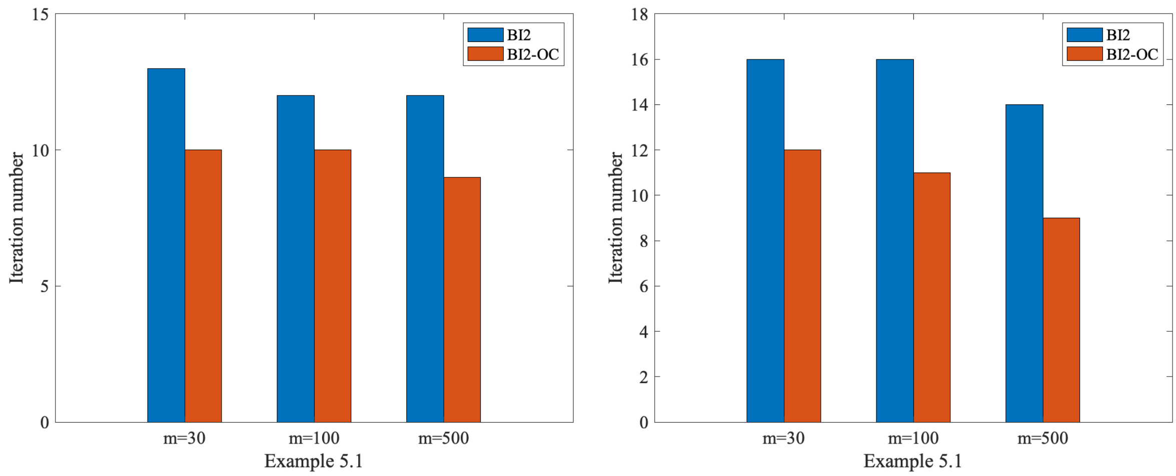

Figure 1: Comparison of iteration number with the methods BI1, BI1-OC (left), and BI2, BI2-OC (right)

Table 1: Numerical results for with $\epsilon = 0.95, \gamma = 1.8683$

Iteration methods	$m = 30$		$m = 100$		$m = 500$	
	Residual	Time	Residual	Time	Residual	Time
BI1	9.73E-14	0.00528	9.30E-13	0.01326	9.31E-13	0.16213
BI1-OC	1.26E-13	0.00151	1.25E-13	0.00882	2.32E-12	0.12486
BI2	4.05E-13	0.00350	4.06E-13	0.01760	2.70E-12	0.27946
BI2-OC	4.75E-14	0.00193	5.95E-13	0.00726	1.69E-13	0.16732

References

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