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On Newton Method for the Minimal Positive Solution of a System of Multi-Variable Nonlinear Matrix Equations

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In this study, we consider the minimal positive solution of the following system of the multi-variable nonlinear matrix equations that can be expressed in the form

$$\begin{cases} A_{1,n}X_1^n + A_{1,n-1}X_2^{n-1} + \cdots + A_{1,2}X_{n-1}^2 + A_{1,1}X_n + A_{1,0} = 0, \\ A_{2,n}X_2^n + A_{2,n-1}X_3^{n-1} + \cdots + A_{2,2}X_n^2 + A_{2,1}X_1 + A_{2,0} = 0, \\ \vdots \\ A_{n,n}X_n^n + A_{n,n-1}X_1^{n-1} + \cdots + A_{2,2}X_{n-2}^2 + A_{n,1}X_{n-1} + A_{n,0} = 0 \end{cases} \tag{1}$$

where $X_i \in \mathbb{R}^{p\times p}$ are unknown matrices, $A_{i,j} \in \mathbb{R}^{p\times p}$ for $i = 1, 2, \ldots, n$ and $j = 0, 1, \ldots, n$. We give the following assumptions on the coefficient matrices of the system (1):

For $i = 1, 2, \ldots, n$ and $j = 2, 3, \ldots, n$,
$A_{i,j}$ is a positive matrix or a nonnegative irreducible matrix,
$-A_{i,1}$ is nonsingular M-matrix,
$A_{i,0}$ is a positive matrix.

For $j = 0, 1, \ldots, n$, set the coefficient matrices A_j, unknown matrix Y and the permutation matrix P, then the system (1) can be equivalently reformulated as

$$\begin{aligned} F(Y) &= A_nY^n + A_{n-1}P^\top Y^{n-1}P + \cdots + A_1(P^\top)^{n-1}YP^{n-1} + A_0 \\ &= \sum_{j=0}^{n} A_j(P^\top)^{n-j}Y^jP^{n-j} = 0. \end{aligned} \tag{2}$$

Newton's iteration for solving equation (2) can be stated as

$$\begin{cases} D_{Y_i}(H_i) = -F(Y_i), \\ Y_{i+1} = Y_i + H_i. \end{cases} \quad i = 1, 2, \ldots$$

$$\text{where } D_Y(H) = \sum_{p=1}^{n} \left(A_p \left(P^\top\right)^{n-p} \left(\sum_{q=0}^{p-1} Y^q H Y^{p-q-1} \right) (P)^{n-p} \right) \tag{3}$$

Note that the matrices in equation (3) is of size $np \times np$, which implies that the computation cost of the classical Newton's iteration is expensive if n and p are large. To reduce the computation cost, we propose the modified Newton's iteration as follows:

$$\begin{cases} A_{1,n}\Gamma_n^{(1,i)}(H_{1,i}) + A_{1,n-1}\Gamma_{n-1}^{(2,i)}(H_{2,i}) + \cdots + A_{1,1}H_{n,i} = -F_1(X_{1,i}, \ldots, X_{n,i}), \\ A_{2,n}\Gamma_n^{(2,i)}(H_{2,i}) + A_{2,n-1}\Gamma_{n-1}^{(3,i)}(H_{3,i}) + \cdots + A_{2,1}H_{1,i} = -F_2(X_{1,i}, \ldots, X_{n,i}), \\ \vdots \\ A_{n,n}\Gamma_n^{(n,i)}(H_{n,i}) + A_{n,n-1}\Gamma_{n-1}^{(1,i)}(H_{1,i}) + \cdots + A_{n,1}H_{n-1,i} = -F_n(X_{1,i}, \ldots, X_{n,i}), \\ X_{1,i+1} = X_{1,i} + H_{1,i}, \\ X_{2,i+1} = X_{2,i} + H_{2,i}, \\ \vdots \qquad\qquad i = 1, 2, \ldots \\ X_{n,i+1} = X_{n,i} + H_{n,i}, \\ \text{where } \Gamma_k^{(j,i)}(H) = \sum_{p=1}^{k} X_{j,i}^{p-1} H X_{j,i}^{k-p} \text{ for } j = 1, 2, \ldots, n. \end{cases} \tag{4}$$

Set $X_{j,0} = 0$ for $j = 1, 2, \ldots, n$, we prove that the sequences $\{X_{j,i}\}$ generated by (4) converge to the minimal positive solution of system (1). And some numerical experiments are given to show the efficiency of the modified Newton's iteration in calculation time and memory.

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