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Elementwise accurate doubling algorithm for shifted M-matrix algebraic Riccati equations

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We consider the nonsymmetric algebraic Riccati equation (NARE)

$$XBX - XA - DX - C = 0, \tag{1}$$

where A, B, C, D are real matrices of sizes $n \times n$, $n \times m$, $m \times n$, $n \times n$, respectively. We focus on the case in which the matrix

$$M = \begin{bmatrix} A & -B \\ C & D \end{bmatrix},$$

is an irreducible singular M-matrix. The problem of finding the minimal non negative solution of such Riccati equations arises in applied probability, transportation theory, fluid queues.

A *structure-preserving doubling algorithm (SDA)* for computing the minimal solution of (1) has been proposed in [3]. This iterative method relies on the fact that the problem of solving (1) can be reduced to the computation of certain invariant subspaces of the matrix $H = \begin{bmatrix} I_n & 0 \\ 0 & -I_m \end{bmatrix} M$, and its convergence properties are connected with a quotient involving the eigenvalues of H.

In [2] Guo et al. studied the doubling algorithm in the case where M is an irreducible singular M-matrix, and in order to speed up the convergence proposed a shift technique to move one zero eigenvalue of H to a positive real number. This approach modifies the equation (1) introducing a shifted equation that shares with (1) the solution, and leads to a reduction of the quotient that controls the convergence so as to produce a dramatic decrease of the number of steps of the algorithm.

In the case where M is a non singular M-matrix or an an irreducible singular M-matrix, algorithms computing the minimal non negative solution of (1) with high elementwise relative accuracy have been proposed in [6], [4], [5] . The general approach is based on the idea that a non singular M-matrix can be inverted by the GTH-like algorithm [1], that consists in a modification of the standard Gaussian elimination in a cancellation-free fashion, when a *triplet representation* of the matrix is known. A triplet representation of A is a triple (P, u, v) such that $P \geq 0$, $u > 0$, $v \geq 0$ with P matrix with null diagonal entries and $A = D - P$, where D is diagonal, and $Au = v$.

Unfortunately, the shifted matrix $\hat{M}$ constructed in [2] in general is no longer a M-matrix, so the known elementwise accurate algorithms can not be applied directly together with the shift technique in order to improve the accuracy and also accelerate the convergence.

We present an elementwise accurate algorithm using the shift technique for the computation of the minimal non negative solution of (1), when M in an irreducible singular M-matrix.

We propose the idea of *delayed shift* and some results that guarantee the applicability and the convergence of structured doubling algorithm based only on the properties of the matrix of the initial setup of doubling algorithm instead of matrix M or $\hat{M}$. We provide a componentwise error analysis for the algorithm and we also show some numerical experiments that illustrate the advantage in terms of accuracy and convergence speed.

References

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