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A distance formula for tuples of doubly commuting matrices

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For a tuple of operators $\boldsymbol{A} = (A_1, \dots, A_d)$, $\operatorname{dist}(\boldsymbol{A}, \mathbb{C}^d\, \boldsymbol{I})$ is defined as $\min_{\boldsymbol{z}\in\mathbb{C}^d} \|\, \boldsymbol{A} - \boldsymbol{z}\, \boldsymbol{I}\, \|$ and $\operatorname{var}_x(\boldsymbol{A})$ as $\|\, \boldsymbol{A}\, x\|^2 - \sum_{j=1}^d \left|\langle x|A_j x\rangle\right|^2$. Ming [3] showed that if $\boldsymbol{A}$ is a tuple of commuting normal operators on a Hilbert space $\mathcal{H}$, then

$$\sup_{\|x\|=1} \operatorname{var}_x(\boldsymbol{A}) = R_{\boldsymbol{A}}^2, \tag{1}$$

where $R_{\boldsymbol{A}}$ is the radius of the smallest disc containing the Taylor spectrum of $\boldsymbol{A}$. We have $R_{\boldsymbol{A}} = \operatorname{dist}(\boldsymbol{A}, \mathbb{C}^d \boldsymbol{I})$.

We will state the idea of the proof of the following. For tuples of doubly commuting matrices, we have

$$\operatorname{dist}(\boldsymbol{A}, \mathbb{C}^d\, \boldsymbol{I})^2 = \sup_{\|x\|=1} \operatorname{var}_x(\boldsymbol{A}).$$

The main facts we will use for the proof will be the normal form for a collection of doubly commuting matrices proved in the main theorem of [1] and the idea of the proof of Proposition 4 of [2].

References

[1] V. Bolotnikov, L. Rodman, *Normal forms and joint numerical ranges of doubly commuting matrices*, Linear Algebra Appl. 301 (1999) 187–194.

[2] A. T. Dash, *Tensor products and joint numerical range*, Proc. Amer. Math. Soc. 40 (1973) 521–526.

[3] F. Ming, *Garske's inequality for an n-tuple of operators*, Integral Equations Operator Theory 14 (1991), 787–793.