Matrix Equations and Tensor Techniques
IX
Perugia, September 9-10, 2021

Solution of the matrix equation for geodesics associated with the Riemannian metric on the space of positive-definite matrices based on the power potential

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In this work, we give a closed-form expression for the nonlinear second-order differential equation of geodesic curves associated with the Riemannian metric given by the Hessian of the power potential function on $\mathcal{P}_n$, the space of positive-definite matrices of order n. The β-power potential function on $\mathcal{P}_n$ is [2]

$$\Phi_\beta(X) = \frac{1-(\det X)^\beta}{\beta}, \quad \beta \neq 0. \tag{1}$$

It generalizes the logarithmic potential in the sense that $\lim_{\beta\to 0} \Phi_\beta(X) = -\ln\det(X)$.

For $\beta < \frac{1}{n}$, the Hessian of (1) is positive definite, and hence it provides at each point $X \in \mathcal{P}_n$ a one-parameter family of Riemannian metrics on $\mathcal{P}(n)$ given by

$$g_{\beta,X}(U,V) := (\det X^\beta)\big(\operatorname{tr}(X^{-1}UX^{-1}V) - \beta\operatorname{tr}(X^{-1}U)\operatorname{tr}(X^{-1}V)\big), \tag{2}$$

where U and V are points of the tangent space to $\mathcal{P}_n$ at X, identified as usual with the space of symmetric matrices of order n.

A geodesic curve $\{X(t),\ t \in [0,1]\}$ with respect to the Riemannian metric (2) satisfies the second-order matrix differential equation

$$\frac{d}{dt}\left(\frac{\partial g_{\beta,X}(X',X')}{\partial X'}\right) - \frac{\partial g_{\beta,X}(X',X')}{\partial X} = O_n. \tag{3}$$

Theorem 1 ([1])**.** *Let $X : [0,1] \to \mathcal{P}(n)$ be a smooth geodesic on $\mathcal{P}(n)$ equipped with the Riemannian metric (2). Then, by introducing the matrix function $G(t) = X^{-1}(t)X'(t)$, the second-order ODE (3) can be written as the decoupled first-order system for X and G:*

$$G' = \frac{\beta}{2(1-n\beta)}\left(\operatorname{tr}(G^2) - \beta\operatorname{tr}^2(G)\right)I - \beta\operatorname{tr}(G)G, \tag{4a}$$

$$X' = XG. \tag{4b}$$

It is worthy to note that (4.a) is a nonlinear (quadratic) ODE for $G(t)$. Once $G(t)$ is obtained, the linear first-order ODE (4.b) can be solved for $X(t)$.

We show that, under some conditions on β, there exists a unique geodesic curve for the metric (2) joining two positive-definite matrices A and B and we provide an explicit expression for this geodesic.

Before we state our main result, let us define the following measure of linear independence between two symmetric positive definite matrices A and B

$$\gamma_\beta(A,B) := \frac{|\beta|\delta(\det(A)^{-1/n}A, \det(B)^{-1/n}B)}{2\sqrt{1/n-\beta}}, \tag{5}$$

where $\delta(\cdot,\cdot)$ is the Riemannian distance on $\mathcal{P}_n$ given for any two matrices $M, N \in \mathcal{P}_n$ by $\delta(M,N) := \left(\sum_{i=1}^n \ln^2 \lambda_i\right)^{1/2}$, with $\lambda_1,\dots,\lambda_n$ are the eigenvalues of $M^{-1}N$.

Theorem 2. *If $A, B \in \mathcal{P}(n)$ are linearly independent, set $d := \delta(\det(A)^{-1/n}A, \det(B)^{-1/n}B)$ and*

$$\beta_1 := -\pi\frac{\sqrt{\pi^2n^2+4nd^2}+\pi n}{2nd^2}, \quad \beta_2 := \pi\frac{\sqrt{\pi^2n^2+4nd^2}-\pi n}{2nd^2}.$$

Then, for $\beta \in (\beta_1, 0)\cup(0,\beta_2)$, there exists a unique geodesic joining A and B given by

$$G_\beta(A,B,t) = \eta(t)A(A^{-1}B)^{\alpha(t)}, \quad t\in[0,1], \tag{6}$$

where

$$\alpha(t) = \frac{1}{\gamma}\arctan\Big(\frac{t\sigma\sin\gamma}{1-t+t\sigma\cos\gamma}\Big), \quad \eta(t) = \Big(\frac{(1-t)^2+2t(1-t)\sigma\cos\gamma+t^2\sigma^2}{\sigma^{2\alpha(t)}}\Big)^{\frac{1}{n\beta}},$$

with $\sigma = \det(A^{-1}B)^{\beta/2}$ and $\gamma := \gamma_\beta(A,B)$.

The geodesic curve (6) has an exponential part, similar to that of the geometric mean, but with exponent $\alpha(t)$; and a scalar power part, $\eta(t)$, which reduces to the weighted $\frac{2}{n\beta}$-power mean when $\gamma = 0$.

When β goes to 0 then (6) becomes the matrix geometric mean

$$\lim_{\beta\to 0} G_\beta(A,B,t) = G_0(A,B,t) := A(A^{-1}B)^t.$$

Furthermore, if A and B are linearly dependent matrices in $\mathcal{P}(n)$, then (6) reduces to the matrix $\frac{n\beta}{2}$-power mean

$$G_\beta(A,B,t) = \big((1-t)A^{\frac{n\beta}{2}} + tB^{\frac{n\beta}{2}}\big)^{\frac{2}{n\beta}}.$$

References

[1] N. Chouaieb, B. Iannazzo, M. Moakher, *Geometries on the cone of positive-definite matrices derived from the power potential and their relation to the power means*, to appear in Linear Algebra and its Applications, (2021).

[2] A. Ohara, N. Suda, S. Amari, *Dualistic differential geometry of positive definite matrices and its applications to related problems*, Linear Algebra and its Applications, **247** (1996) pp. 31–53.